Entropy demystified

RSE24

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Entropy demystified…

1) Historical outline
2) Entropy as a measure of information
3) Entropy and probability
4) Entropy in Physics
5) What exactly is temperature?
6) Sampling
7) A funny story

1) Historical outline

Rudolf Clausius (1822-1888)

- **1855 ETH Zürich**
- **1867 University of Würzburg**
- **1869 University of Bonn**

… was one of the first Theoretical Physicists

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ÜBER DEN

ZWEITEN HAUPTSATZ

DER

MECHANISCHEN WÄRMETHEORIE

Ein Vortrag,

gehalten in einer allgemeinen Sitzung der 41. Versammlung
deutscher Naturforscher und Aerzte zu Frankfurt a. M.
am 23. September 1867

von

Rudolph Julius Emmanuel

R. CLAUSIUS,

Professor der Physik an der Universität zu Würzburg.

Entropy = Internal Transformability

- **en = internal**
- **tropḗ = transformation**

$$S$$

Clausius, April 24, 1865

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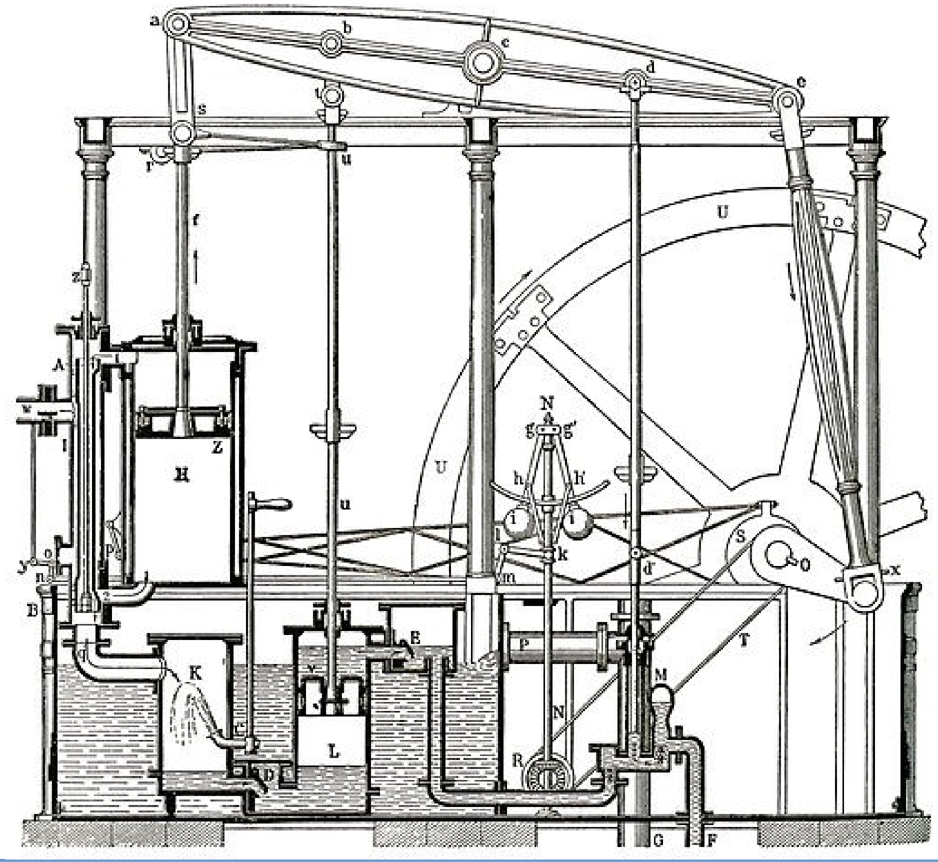


Clausius and the Second Law of Thermodynamics

- The **change** of the entropy S is

$$\Delta S = \frac{\Delta Q}{T}$$

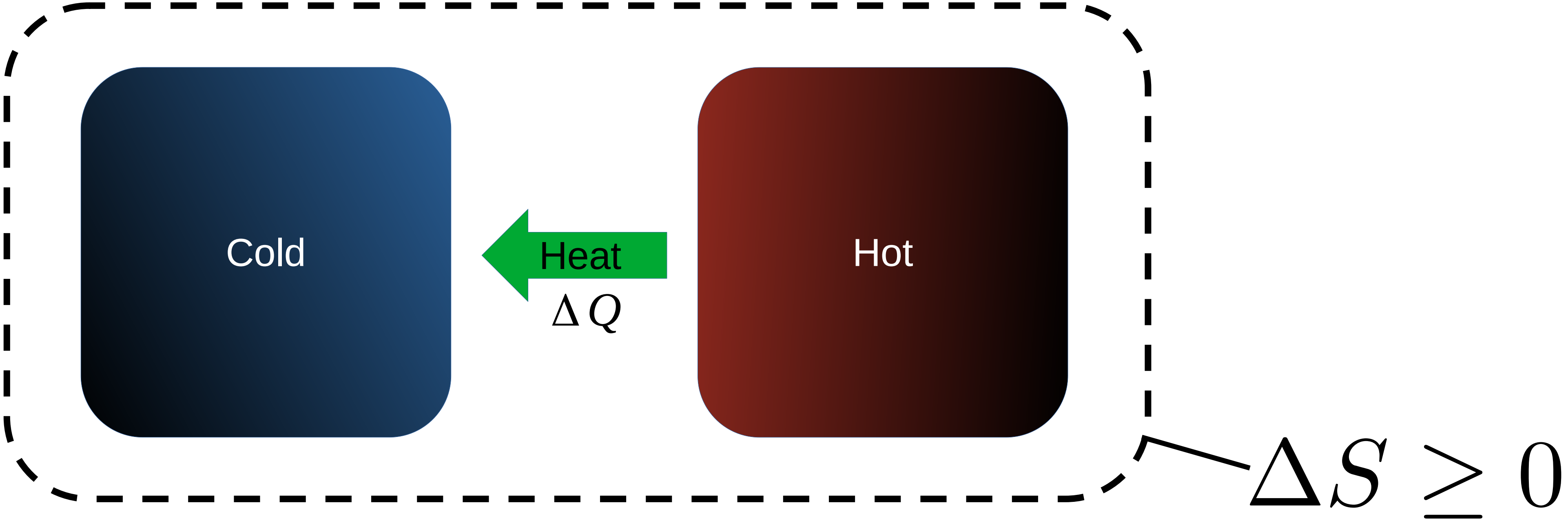
Heat

Temperature

- In isolated systems the entropy can only increase:

$$\Delta S \geq 0$$

The Second Law of Thermodynamics



Original paper by Clausius

44 Abhandlung IX.

man daneben zugleich den anderen, seiner Bedeutung nach einfacheren Begriff der *Energie* anwendet, man die den beiden Hauptsätzen der mechanischen Wärmetheorie entsprechenden Grundgesetze des Weltalls in folgender einfacher Form aussprechen kann:

1) *Die Energie der Welt ist constant.*
2) *Die Entropie der Welt strebt einem Maximum zu.*

1) The Energy of the world is constant.
2) The Entropy of the world tends to a maximum.

The Second Law comes as a surprise, because...

- all mechanical systems are invariant under **time reversal,** meaning that they can also run backward in time.

$$F = ma$$

- but **Entropy increases only with increasing time**.

  This defines the „arrow of time“.

Ludwig Boltzmann (1840-1906)

- 1863 Vienna
- 1869 Graz
- 1873 Vienna (Mathematics)
- 1876 Graz (Experimental Physics)
- 1890 Munich

Co-founder of „atomism“

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Boltzmann’s View

„The question of whether matter is **composed of atoms or continuous** is therefore reduced to whether those properties most accurately represent the observed properties of the matter when the number of particles is assumed to be extraordinarily large.“

Boltzmann's definition of entropy

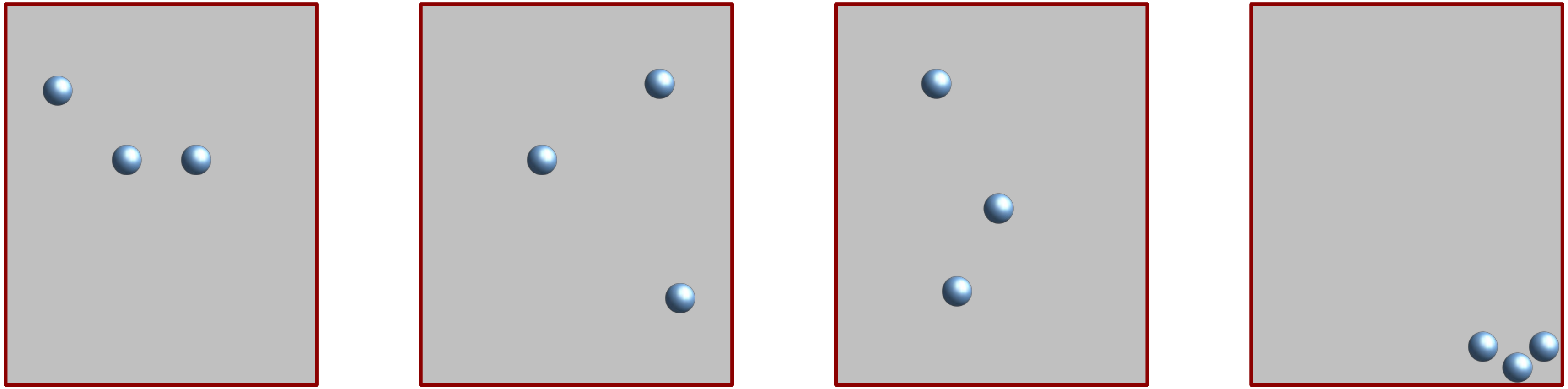
The entropy of an isolated system is

$$S = k_B \log \Omega$$

Boltzmann constant: 1.38×10^{-23} J/K

Number of complexions

What are “complexions”?



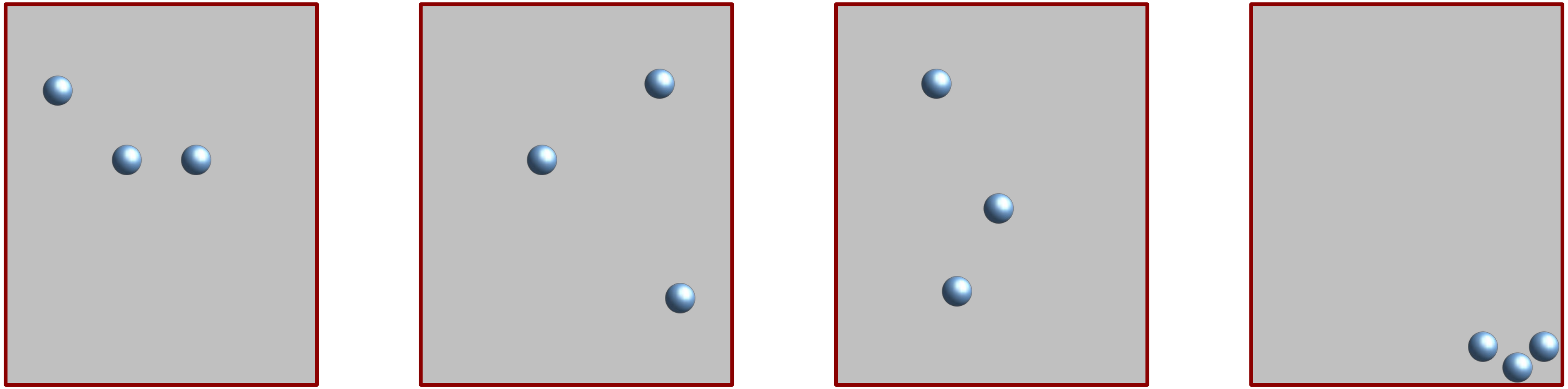
Today we would probably call it “configurations”.

Boltzmann's definition of entropy

$$S = k_B \log \Omega$$

Number of possible configurations
the gas molecules can be in.

What are “complexions”?



This raises the question of how to
“count“ the number of configurations

Claude Shannon (1916-2001)

- Study of mathematics/electrical engineering
- 1941 AT&T Bell Labs
- 1958 MIT

… Father of the information age

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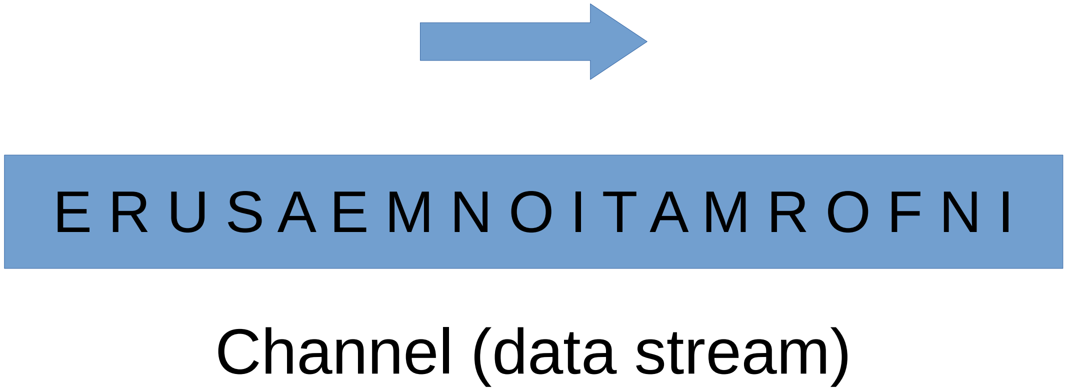
Shannon's most important work

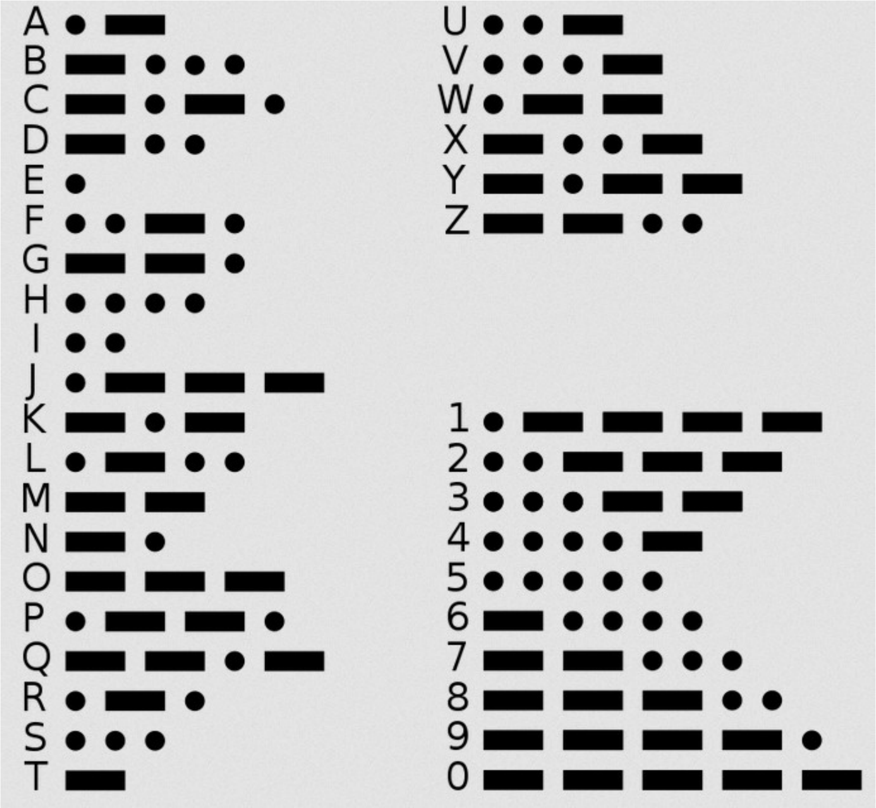
What is the maximum amount of information that can sent through a cable with a given bandwidth?

Shannon's most important work



E	12.7%	
T	9.1%	
A	8.2%	
O	7.5%	
I	7.0%	
N	6.7%	
S	6.3%	
H	6.1%	
R	6.0%	
D	4.3%	

Shannon's most important work



Samuel F. B. Morse
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E	12.7%
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I	7.0%
N	6.7%
S	6.3%
H	6.1%
R	6.0%
D	4.3%

Shannon's most important result

$$H = -\sum_{i=A,B,C,\ldots} p_i \log p_i$$

Average Information Content per character

Sum over all characters

Probability of the characters

Shannon's key result - characteristics

(1) The average information stream is in fact an average:

$$H = -\sum_i p_i \log p_i = \overline{-\log p}$$

$\boxed{-\log p_i \text{ is the information carried by the letter } i}$

Shannon's key result - characteristics

(2) If all characters were **equally likely**, that is, if $p_i = \dfrac{1}{N}$ we would get:

$$H = -\underbrace{\sum_{i=1}^{N} \frac{1}{N}}_{=1} \log \frac{1}{N} = \log N$$

Boltzmann-Entropy

Shannon Information

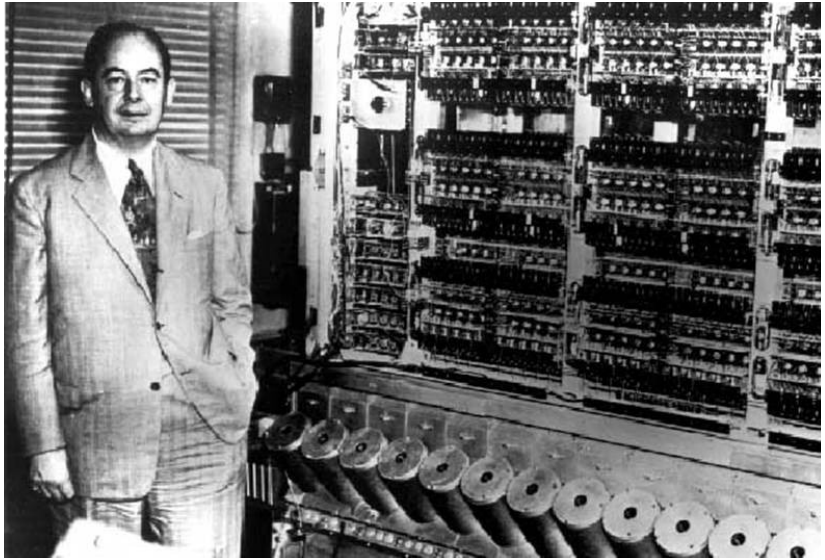
$$S = k_B \log \Omega$$

Number of “complexions”

$$H = \log N$$

Number of characters

Claude Shannon asks John von Neumann: How should I name this quantity?



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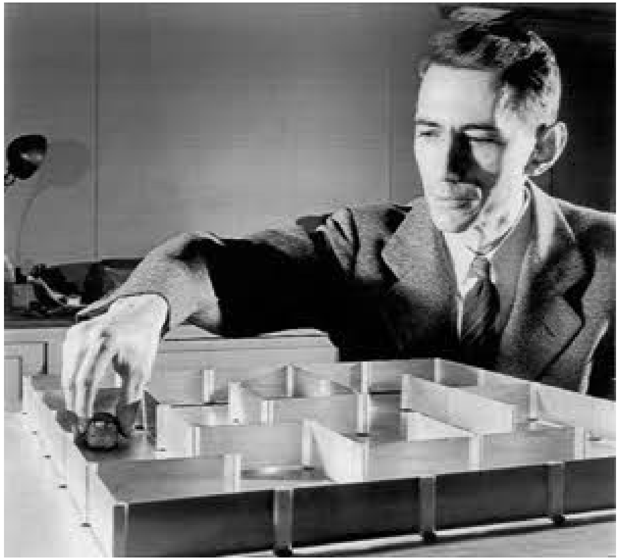


Call it entropy!

No one knows what entropy really is, so in a debate you will always have an advantage.

Puzzle

Heat transfer	Second Law	Number of configurations	Shannon-Information
$$\Delta S = \frac{\Delta Q}{T}$$	$$\Delta S \geq 0$$	$$S = k_B \log \Omega$$	$$H = -\sum_i p_i \log p_i$$ $$\Downarrow$$ $$H = \log N$$

Puzzle

Heat transfer	Second Law	Number of configurations	Shannon-Information
$\Delta S = \frac{\Delta Q}{T}$	$\Delta S \geq 0$	$S = k_B \log \Omega$ ✓	$H = -\sum_i p_i \log p_i$ ⇓ $H = \log N$ ✓

Entropy = Information

2) Entropy = Information

What exactly is information?

Information

Article Talk Read

From Wikipedia, the free encyclopedia

For other uses, see Information (disambiguation).

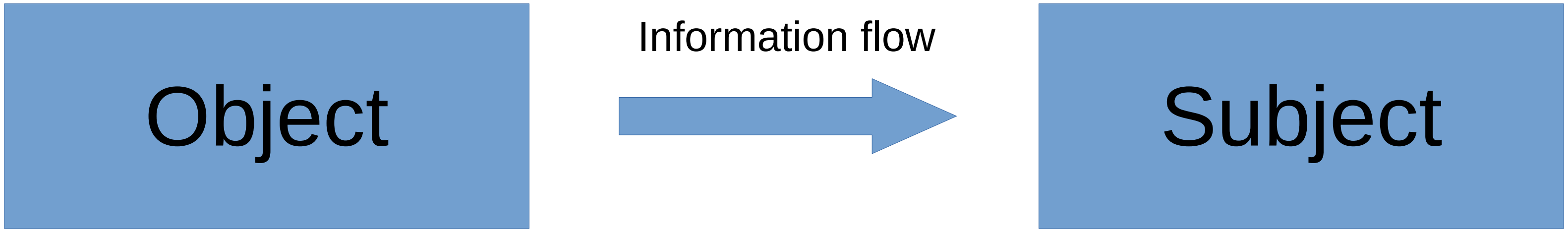
Information is an abstract concept that refers to that which has the power to inform. At the most fundamental level, information pertains to the interpretation (perhaps formally) of that which may be sensed, or their abstractions. Any natural process that is not completely random and any observable pattern in any medium can be said to convey some amount of information. Whereas digital signals and other data use discrete signs to convey information, other phenomena and artifacts such as analogue signals, poems, pictures, music or other sounds, and currents convey information in a more continuous form.[1] Information is not knowledge itself, but the meaning that may be derived from a representation through interpretation.[2]

The concept of *information* is relevant or connected to various concepts,[3] including constraint, communication, control, data, form, education, knowledge, meaning, understanding, mental stimuli, pattern, perception, proposition, representation, and entropy.

Information is often processed iteratively: Data available at one step are processed into information to be interpreted and processed at the next step. For example, in written text each symbol or letter conveys information relevant to the word it is part of, each word conveys information relevant to the phrase it is part of, each phrase conveys information relevant to the sentence it is part of, and so on until at the final step information is interpreted and becomes knowledge in a given domain. In a digital signal, bits may be interpreted into the symbols, letters, numbers, or structures that convey the information available at the next level up. The key characteristic of information is that it is subject to interpretation and processing.

By conveying information, ignorance about an object or issue is reduced.

Flow of information



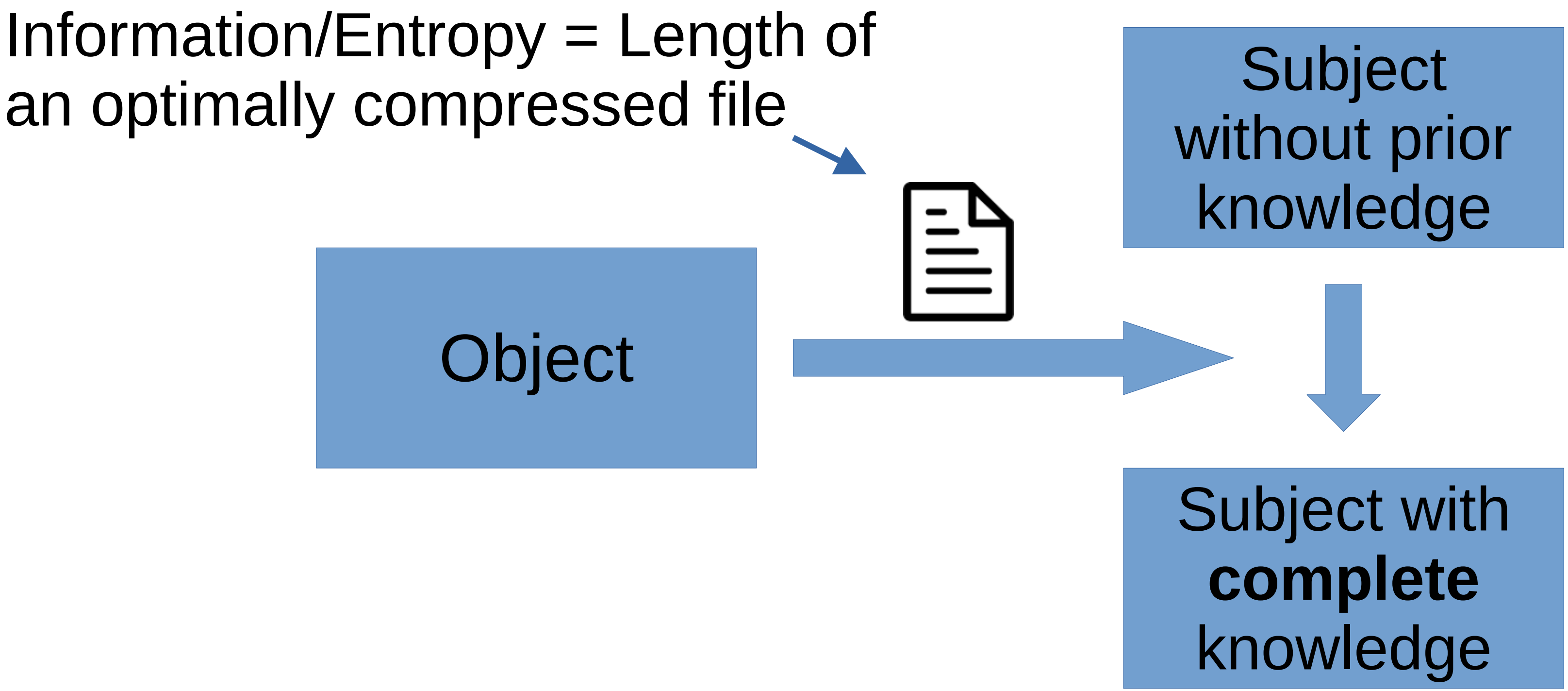
- Physical system
- Fact
- Data set

- Human
- Measurement device
- Interacting system

Verbal definition of information / entropy

The information / entropy of an object is the
minimal size of a data set
that has to be transmitted to
fully describe the object in a given context.

Verbal definition of information



Verbal definition of information

Information is the length of an optimally compressed file that is required to fully describe an object or a fact.

Unit of information: *bit*

Byte	B	1 Byte = 8 bit
Kilobyte	kB	1 kB = 1024 Byte
Megabyte	MB	1 MB = 1024 kB
Gigabyte	GB	1 GB = 1024 MB
Terabyte	TB	1 TB = 1024 GB

Contextuality of entropy / information

The information of an object depends on the chosen **context**.

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- Context: switch position on/off
  Information: 1 bit
- Context: position of all molecules
  Information: huge

Configuration space

- By Ω we denote the set of all possible configurations in a chosen context, called **configuration space**.

$$\Omega = \{0, 1\}$$

- Let |Ω| be the number of all possible configurations

$$H = \log_2 |\Omega|$$

← measured in *bit*

Example: Entropy of a die

Context: number of pips

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$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Cardinality: $|\Omega| = 6$

$$\Rightarrow \; H = \log_2 6 = \frac{\ln 6}{\ln 2} \approx 2.585 \; \textit{bit}$$

Non-integer number of bits ???

Non-integer number of bits ?

Number of dice	Number of configurations	Required number of bits	Bits per die
1	6	3	3
2	36	6	3
3	216	8	2,666
4	1296	11	2,75
5	7776	13	2,6
∞	∞	∞	~ 2,585

Definition of entropy in various disciplines

Computer Science: $$H = \log_2 N$$

Mathematics: $$H = \ln |\Omega|$$

Physics / Chemistry: $$S = k_B \ln \Omega$$

All variants differ only by a prefactor.

How many bits of entropy does a balloon contain?

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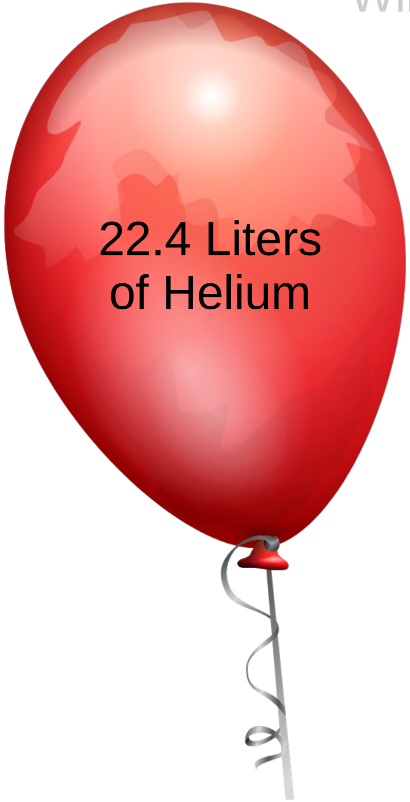
$$S \approx 126\,\mathrm{J/K}$$

chemistrytable.webs.com/enthalpyentropyandgibbs.htm



How many bits of entropy does a balloon contain?

$$S \approx 126\,\mathrm{J/K}$$

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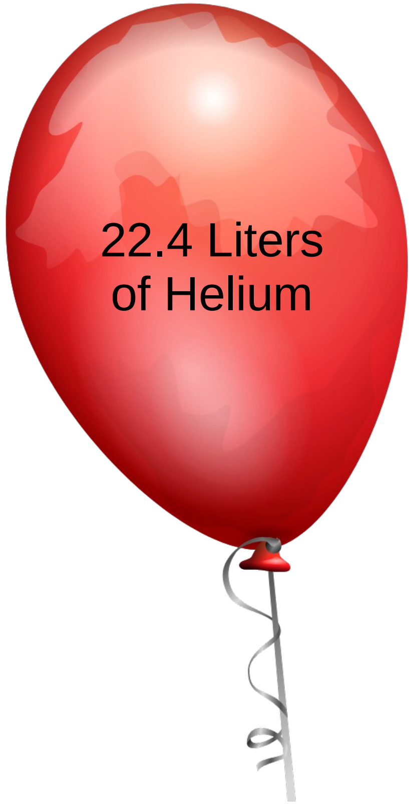
$$H = \frac{1}{k_B \ln 2} S \approx 1.3 \times 10^{25}\,\mathrm{bit}$$



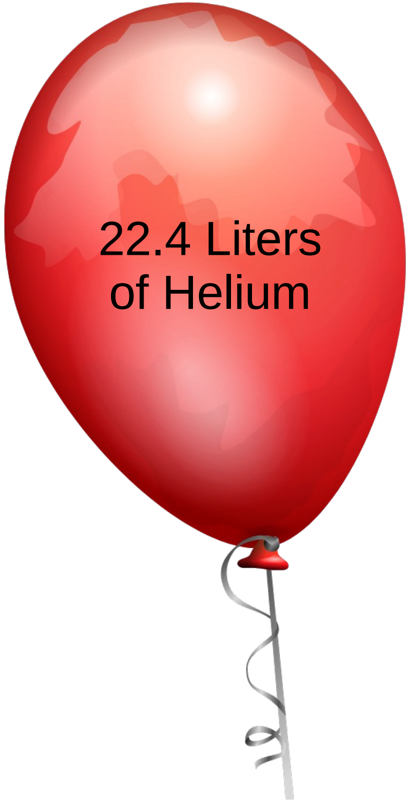
How many bits of entropy does a balloon contain?

$$S \approx 126\,\mathrm{J/K}$$

chemistrytable.webs.com/enthalpyentropyandgibbs.htm

$$H = \frac{1}{k_B \ln 2} S \approx 1.3 \times 10^{25}\,\mathrm{bit}$$

$$\Rightarrow \quad \frac{H}{N_A} \approx 22\,\mathrm{bit}$$



3) Entropy and probability

What exactly is probability?

The train comes with a probability of about 50%

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Probabilities express **partial knowledge**.

Configuration space Ω equipped with probabilities

Context: Arrival of the train

A: in time
B: delayed
C: cancelled

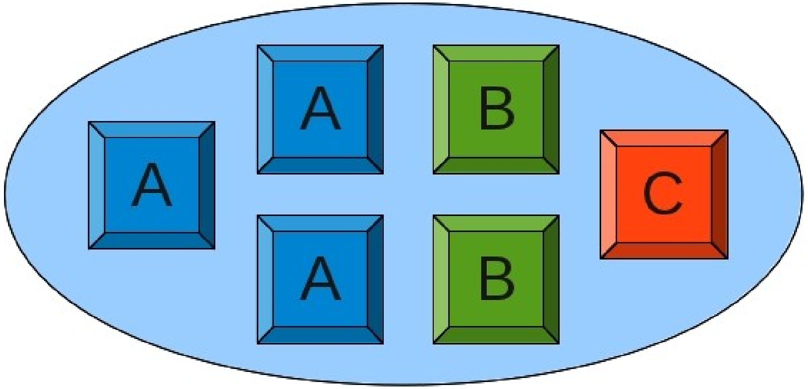
$$\Omega = \{A, B, C\}$$

$$p_A = \frac{1}{2},\ p_B = \frac{1}{3},\ p_C = \frac{1}{6}$$

Entropy of differently probable configurations

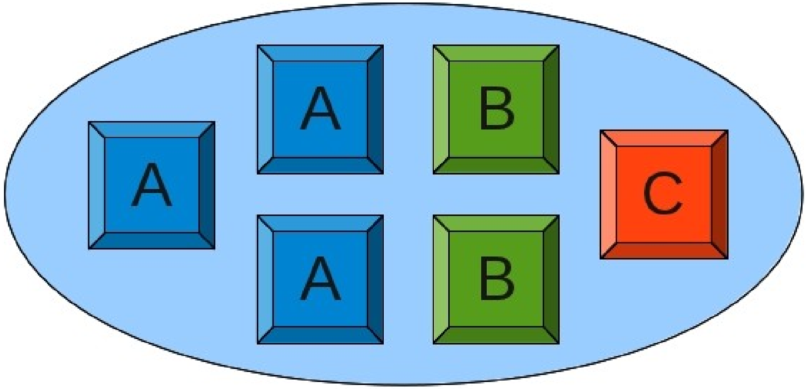
$$\Omega = \{A, B, C\}$$

$$p_A = \frac{1}{2},\ p_B = \frac{1}{3},\ p_C = \frac{1}{6}$$



Trick: Fictitious configuration space in which all elements are equally likely.

Entropy of differently probable configurations

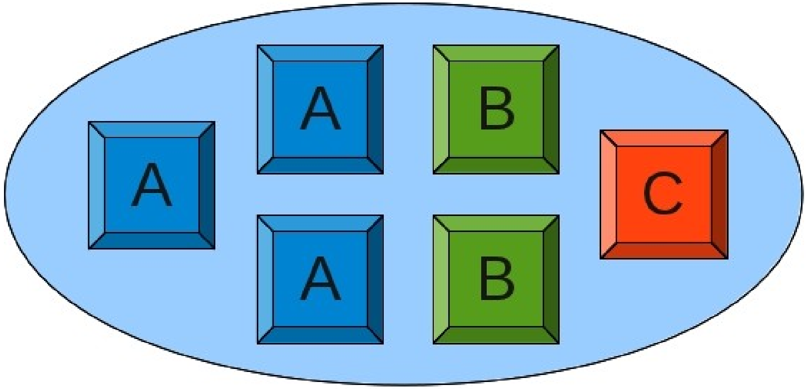


Information needed to specify one of the 6 elements: $H = \log_2 6$

Information needed to specify **A**: $H_A = \log_2 6 - \log_2 3$

Entropy of differently probable configurations

Individual Entropy / Information of the event A:



$$H_A = \log_2 6 - \log_2 3 = \log_2 \frac{6}{3} = \log_2 \frac{1}{p_A} = -\log_2 p_A$$

Entropy of differently probable configurations

Every event $\omega \in \Omega$ with probability $p_\omega \in (0, 1]$

carries an **individual entropy / information**

$$\boxed{H_\omega = -\log_2 p_\omega}$$

Entropy of differently probable configurations

Every event $\omega \in \Omega$ with probability $p_\omega \in (0, 1]$

carries an **individual entropy / information**

$$\boxed{H_\omega = -\log_2 p_\omega}$$

Average: $$\boxed{H = \overline{H_\omega} = \sum_{\omega \in \Omega} p_\omega H_\omega = -\sum_{\omega \in \Omega} p_\omega \log_2 p_\omega}$$

Puzzle

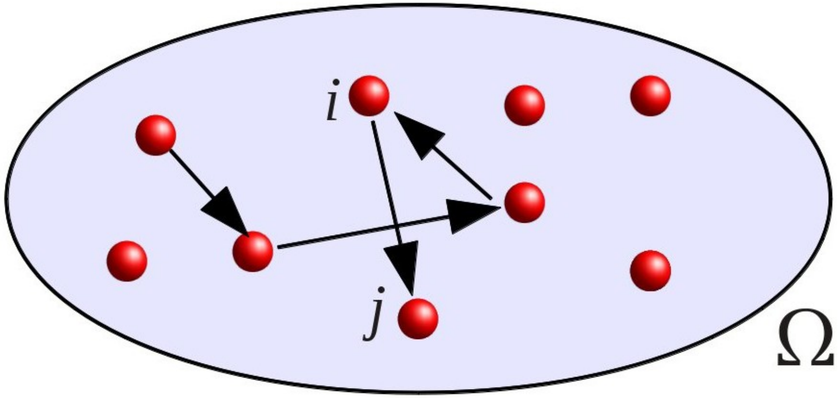
Heat transfer	Second Law	Number of configurations	Shannon-Information
$\Delta S = \frac{\Delta Q}{T}$	$\Delta S \geq 0$	$S = k_B \log \Omega$ ✓	$H = -\sum_i p_i \log p_i$ ✓ ⇓ $H = \log N$ ✓

Entropy = Information

4) Entropy in Physics

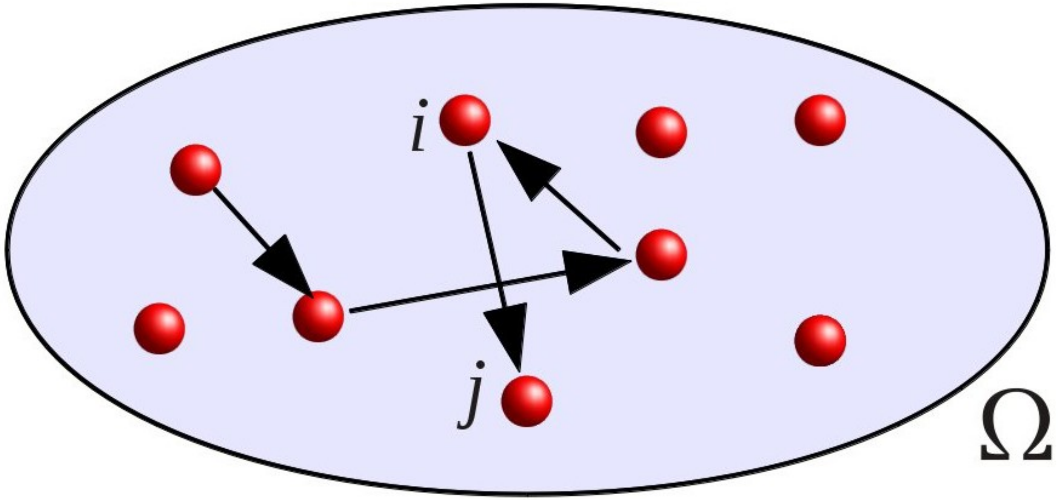
Cartoon of a physical system

Center of Nanoscience, München



Configuration Space

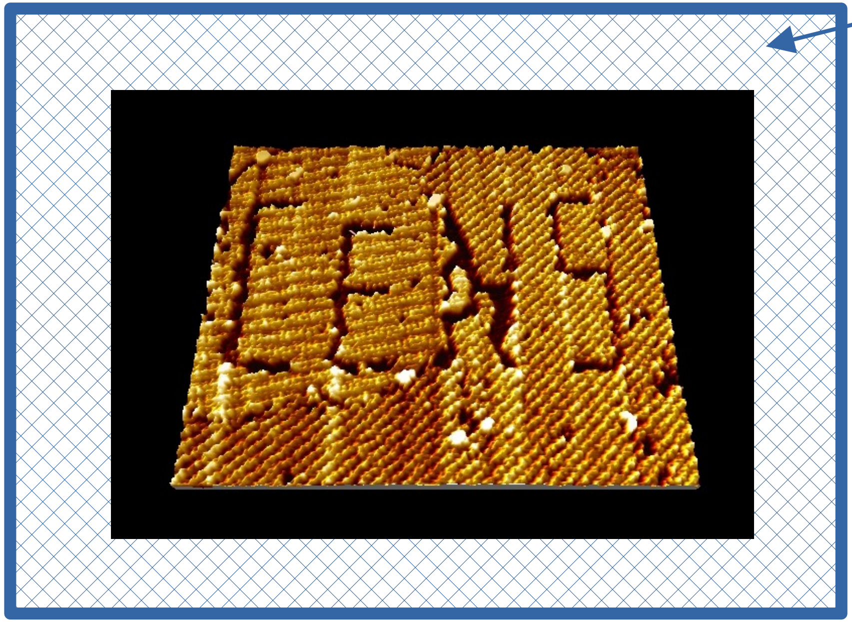
Spontaneous jumps in configuration space



Probabilities per unit time
for spontaneous jumps in Ω:

$$w_{i\to j} \geq 0$$

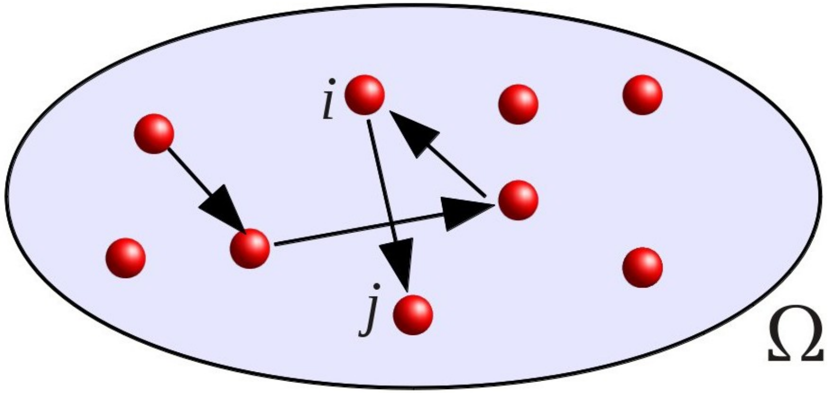
Isolated physical systems: Time reversal symmetry



In **isolated** systems the jump probabilities are symmetric:

$$w_{i \to j} = w_{j \to i}$$

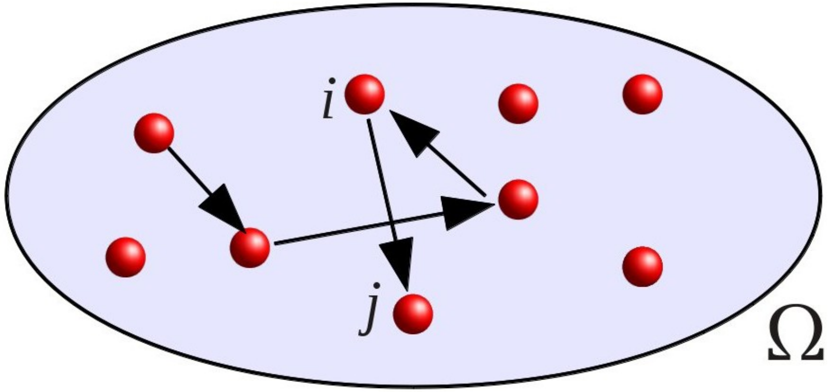
Gibbs' postulate



In **isolated** physical systems, after long equilibratrion time, each configuration is equally likely!

$$\lim_{t \to \infty} p_\omega = \frac{1}{|\Omega|}$$

Second Law of Thermodynamics



The system performs an unbiased random walk in its own configuration space.

$$\lim_{t\to\infty} p_\omega = \frac{1}{|\Omega|}$$

$$\Delta H \geq 0$$

$$H_{\max} = \log_2 |\Omega|$$

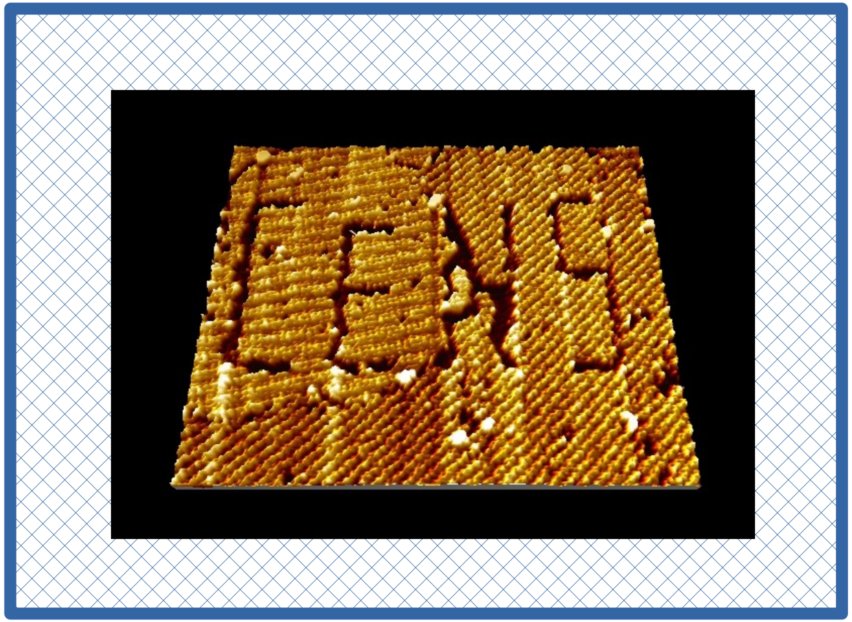
Puzzle

Heat transfer	Second Law	Number of configurations	Shannon-Information
$\Delta S = \frac{\Delta Q}{T}$	$\Delta S \geq 0$ ✓	$S = k_B \log \Omega$ ✓✓	$H = -\sum_i p_i \log p_i$ ✓ $\Downarrow$ $H = \log N$ ✓

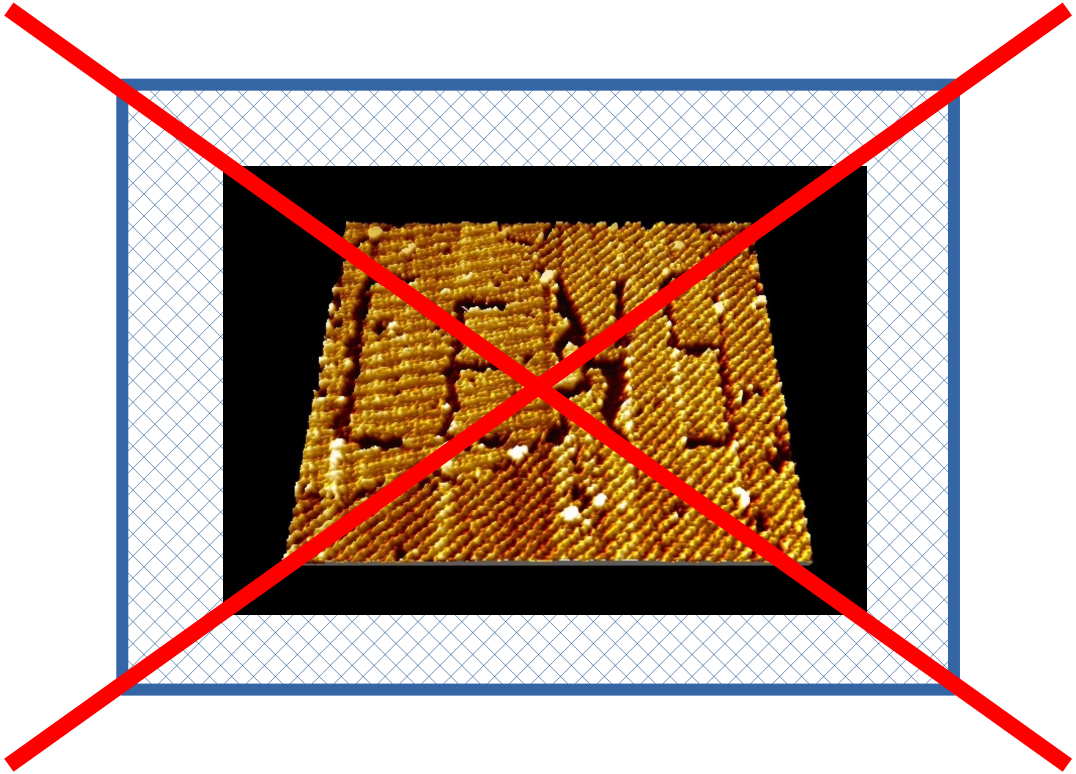
Entropy = Information

5) What exactly is temperature?

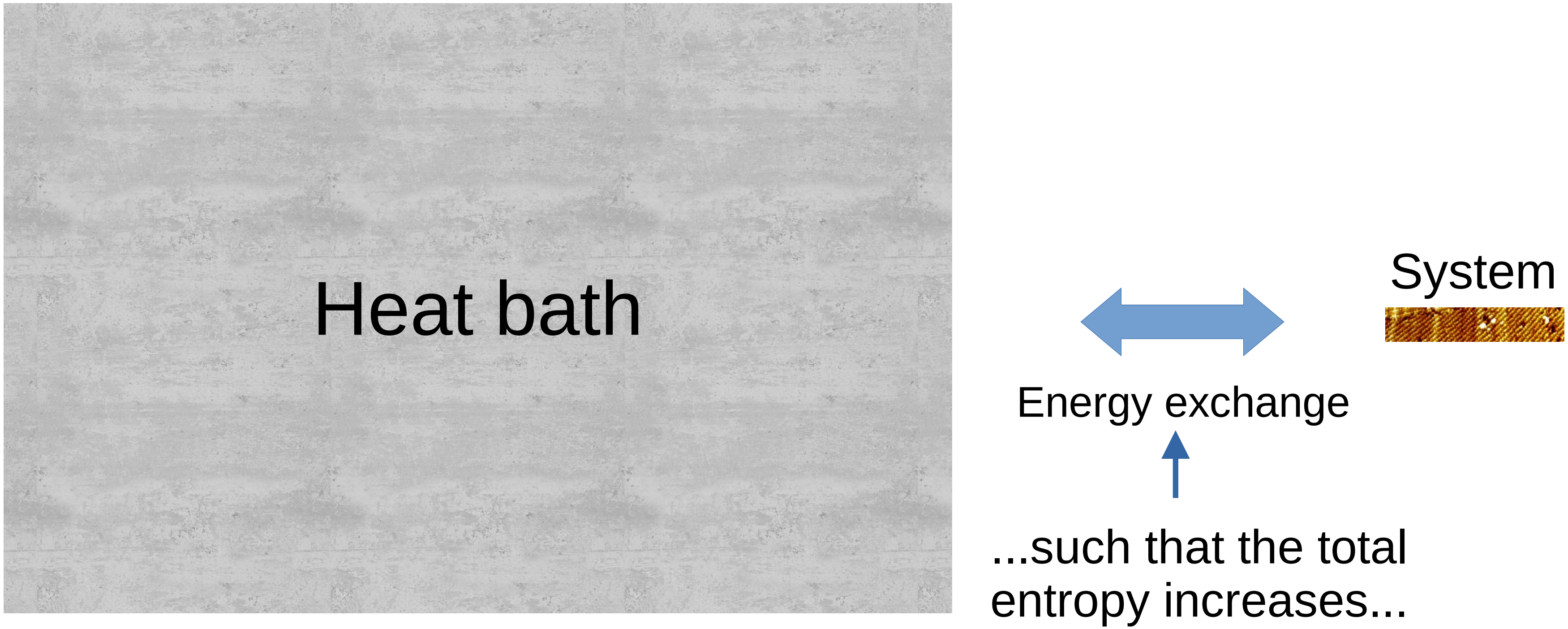
Perfectly isolated physical systems…



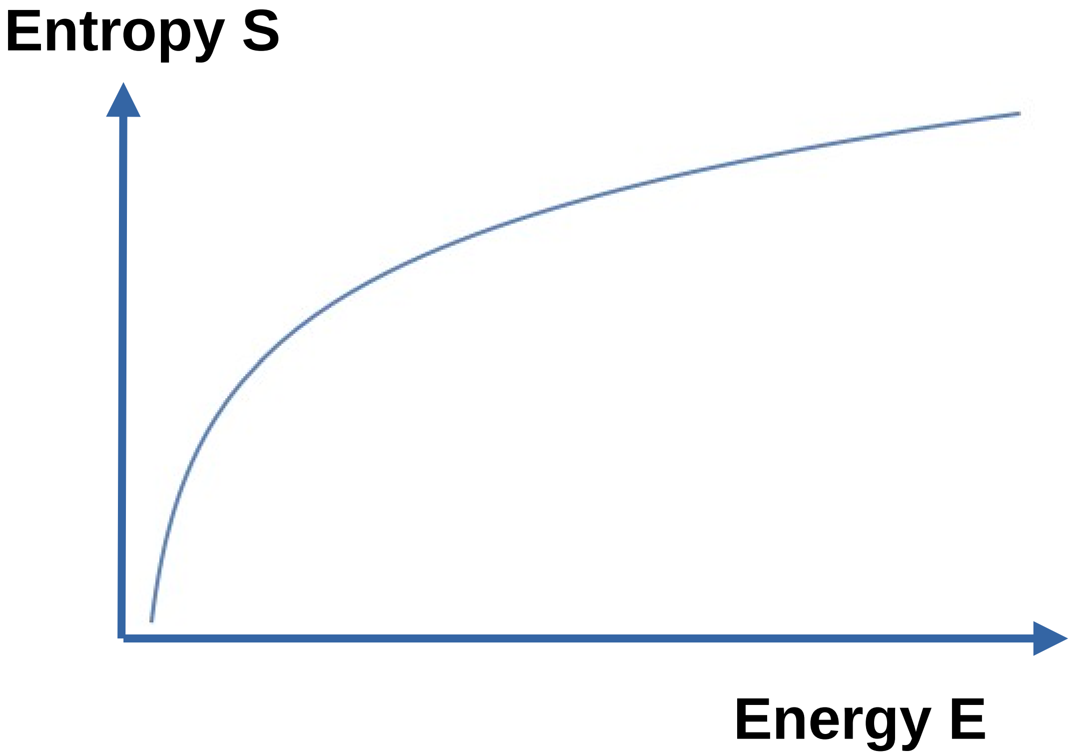
...are unlikely in Nature



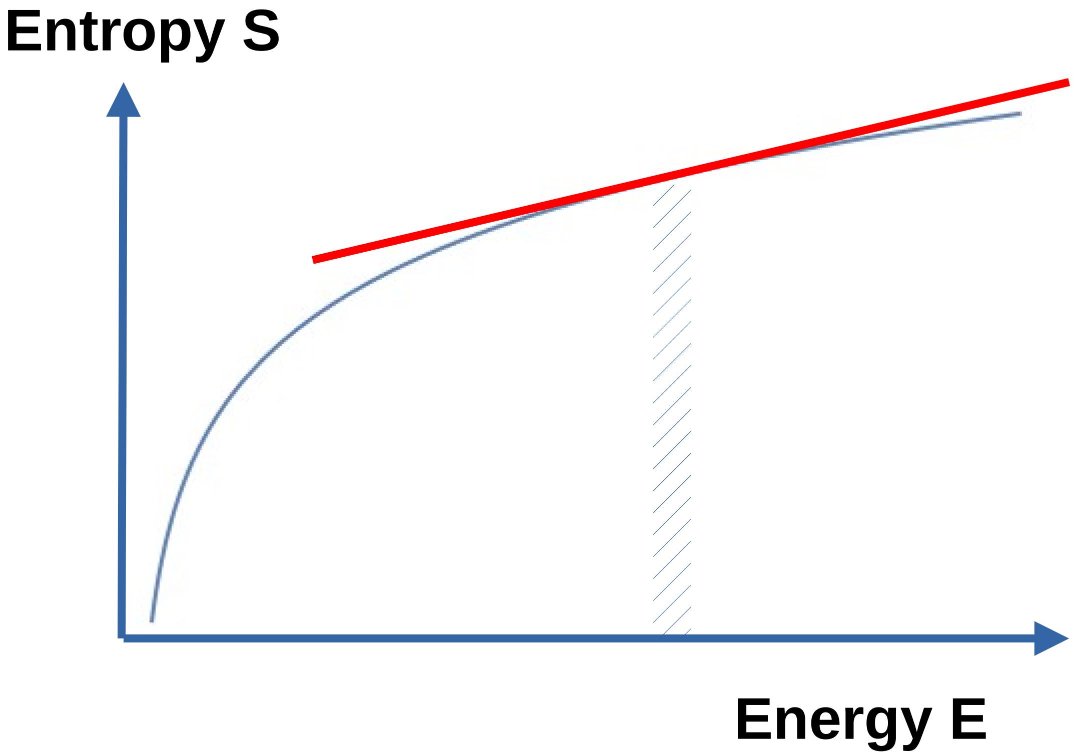
Heat bath = Energy reservoir

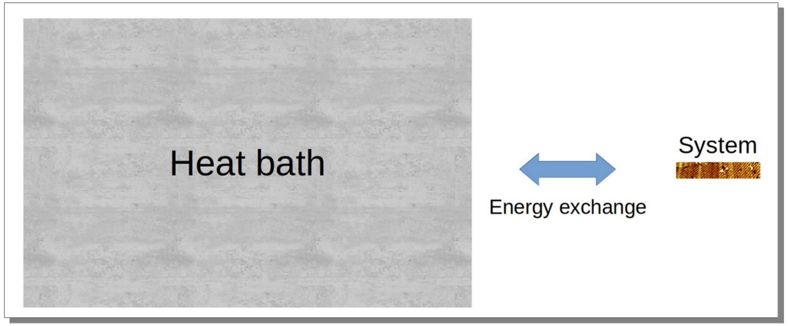


Entropy of a heat bath



Entropy of a heat bath

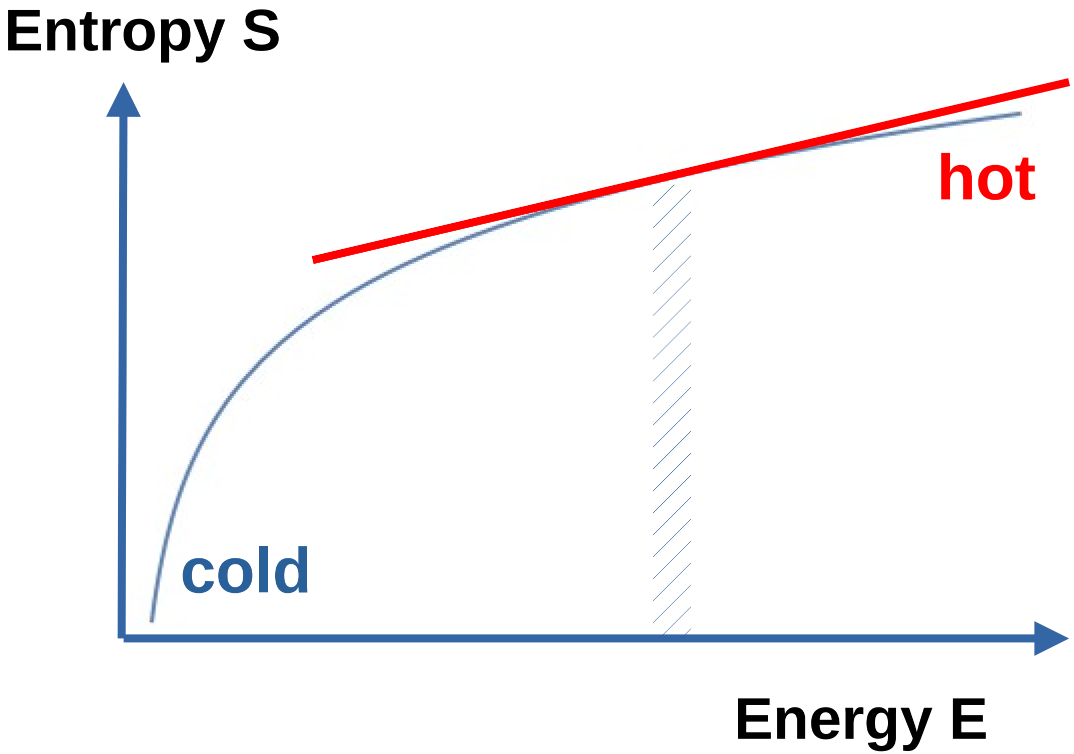


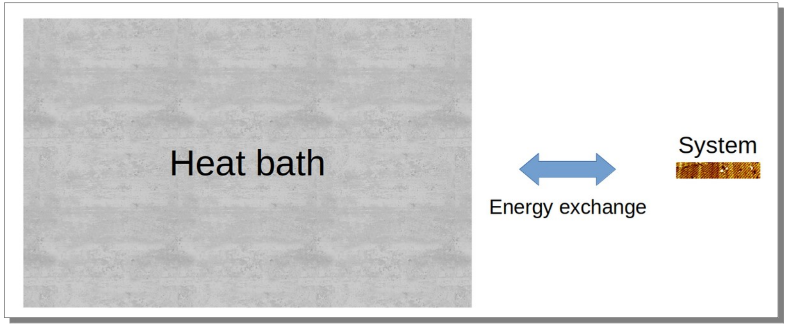


Slope of the tangent

$$\beta = \frac{\Delta S}{\Delta E} = \frac{1}{T}$$

Entropy of a heat bath





Slope of the tangent

$$\beta = \frac{\Delta S}{\Delta E} = \frac{1}{T}$$

Energy is traded for Entropy

$$\text{Price} = \frac{\text{Entropy}}{\text{Energy}} = \frac{1}{T}$$

Currency: Entropy

The “correct“ unit of temperature…

$$\beta = \frac{\Delta S}{\Delta E} = \frac{1}{T}$$

$$[\beta] = \frac{\text{bit}}{\text{Joule}}$$

$$[T] = \frac{\text{Joule}}{\text{bit}}$$

The “correct“ unit of temperature

$$\beta = \frac{\Delta S}{\Delta E} = \frac{1}{T}$$

$$[\beta] = \frac{\text{bit}}{\text{Joule}}$$

$$[T] = \frac{\text{Joule}}{\text{bit}}$$

Computer Science: $H = \log_2 N$

Mathematics: $H = \ln |\Omega|$

Physics / Chemistry: $S = k_B \ln \Omega$

$$k_B = 1.38 \times 10^{-23} \frac{\text{Joule}}{K\ \text{bit}}$$

Room Temperature $\frac{\partial E}{\partial H} = k_B 293\,K \approx 4 \times 10^{-21} \frac{\text{Joule}}{\text{bit}}$

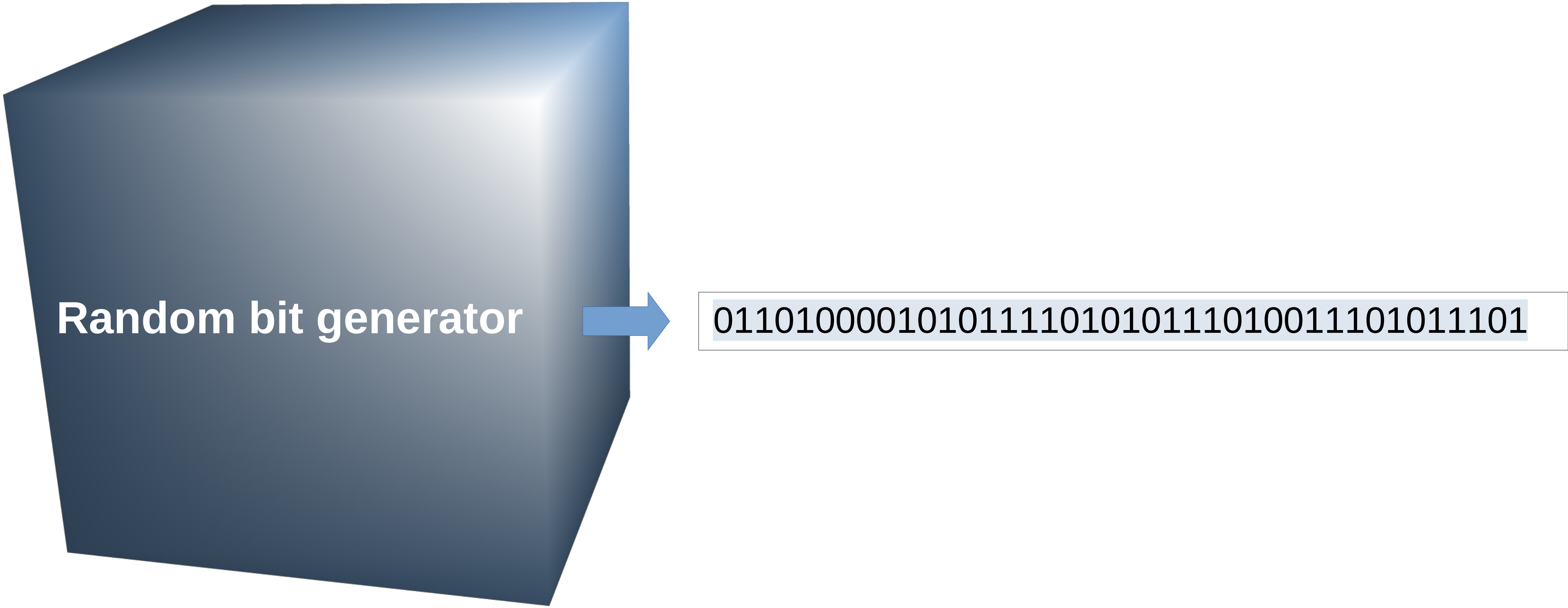
Puzzle

Heat transfer	Second Law	Number of configurations	Shannon-Information
$\Delta S = \frac{\Delta Q}{T}$ ✓	$\Delta S \geq 0$ ✓	$S = k_B \log \Omega$ ✓✓	$H = -\sum_i p_i \log p_i$ ✓ ⇓ $H = \log N$ ✓

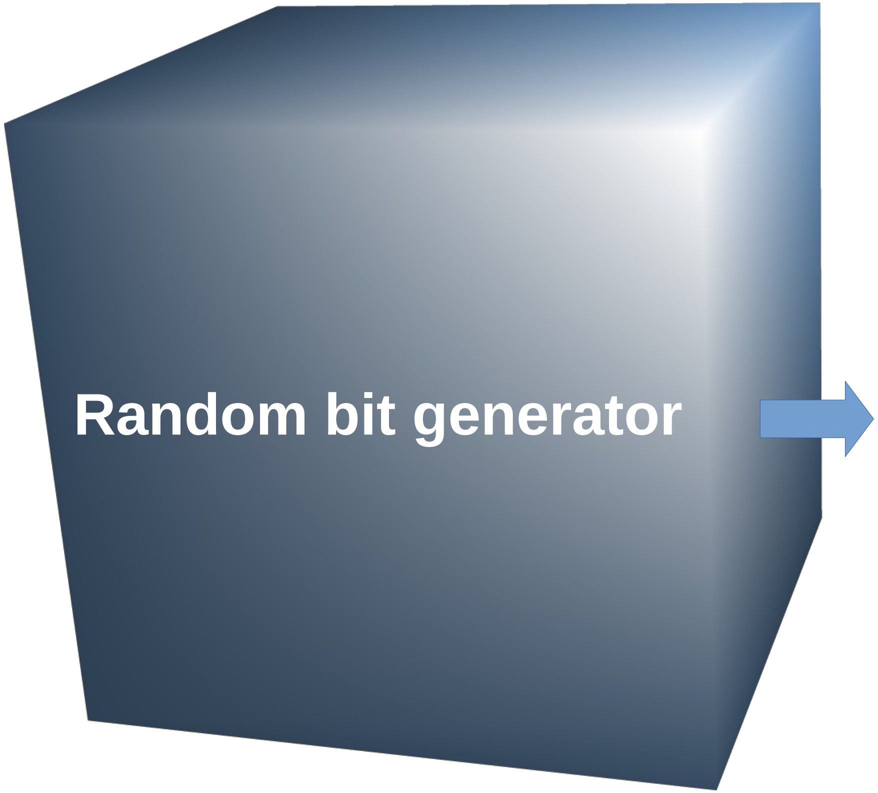
Entropy = Information

6) Sampling

Random data source



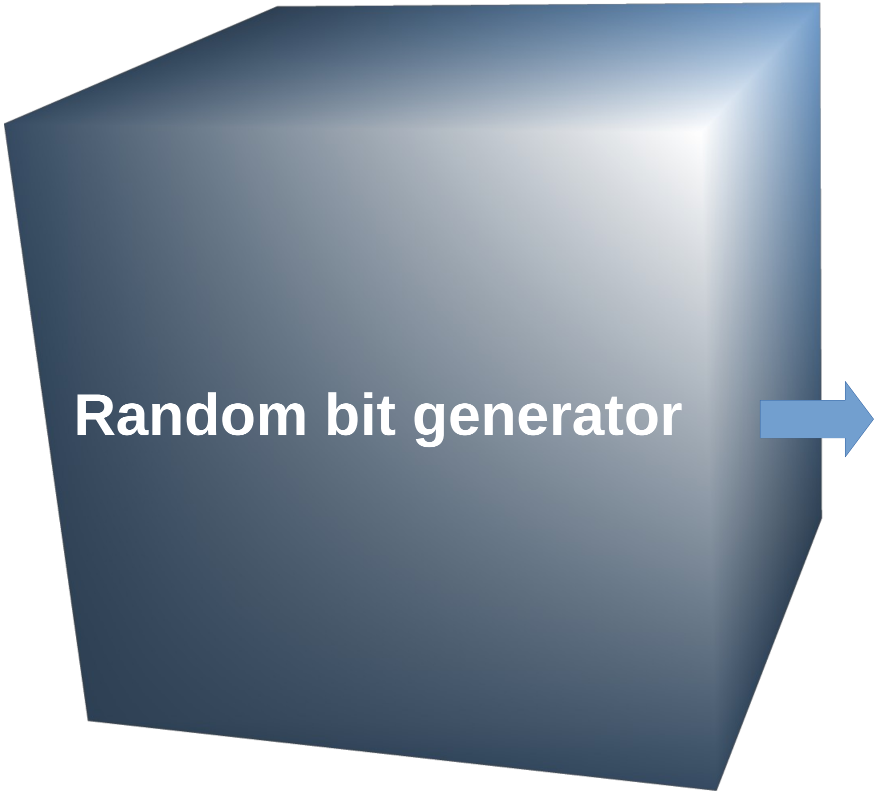
Random data source



We would like to estimate its entropy.

0110100001010111101010111010011101011101

Random data source

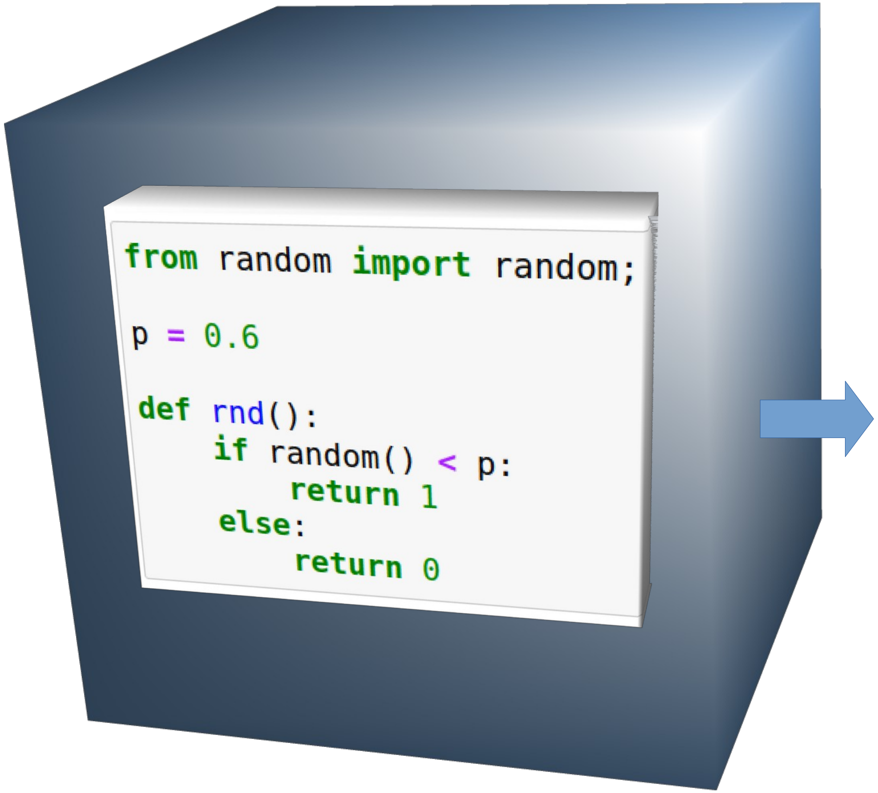


Let p be the probability of a 1
and $1\text{-}p$ the probability of a 0

0110100001010111101010111010011101011101

$$H = -p\log_2(p) - (1-p)\log_2(1-p)$$

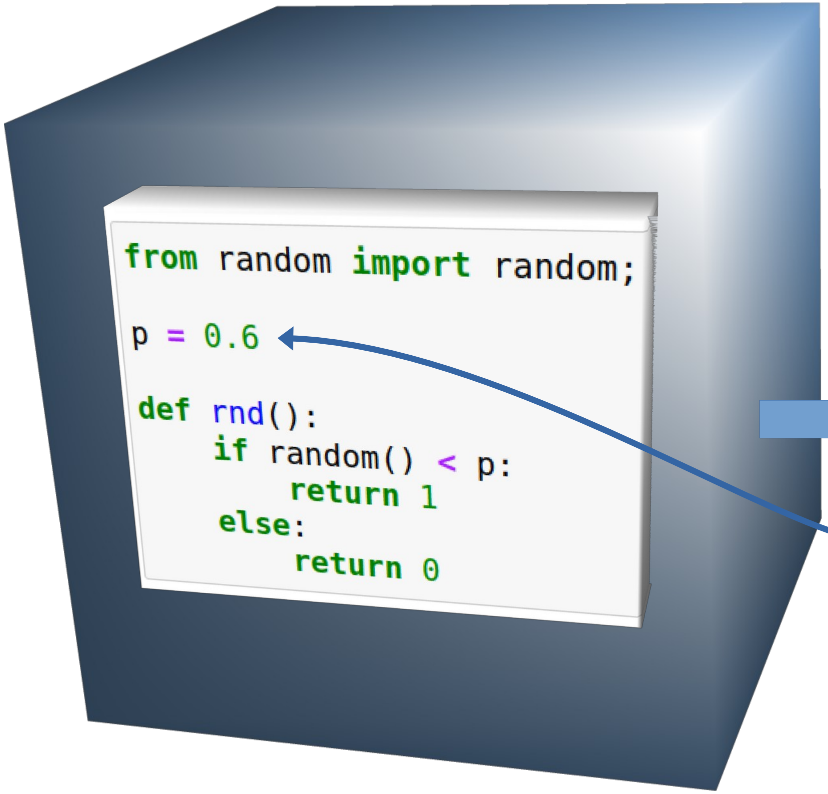
Random data source



0110100001010111101010111010011101011101

$$H = -p\log_2(p) - (1-p)\log_2(1-p)$$

Random data source

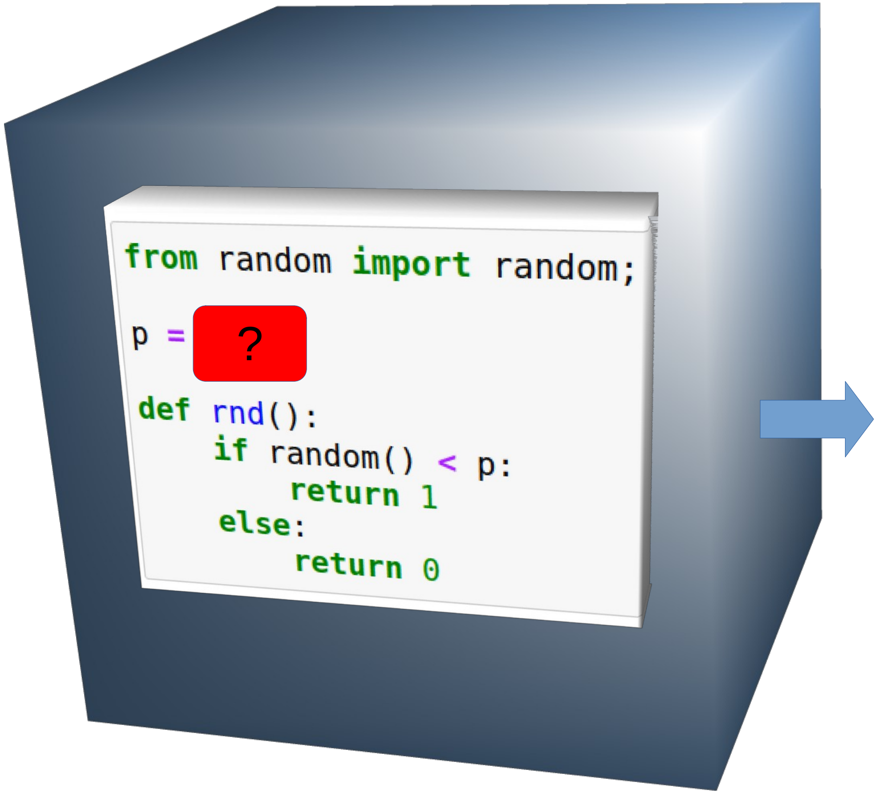


0110100001010111101010111010011101011101

$$H = -p\log_2(p) - (1-p)\log_2(1-p)$$

$$H = 0.97095 \text{ bit}$$

Random data source

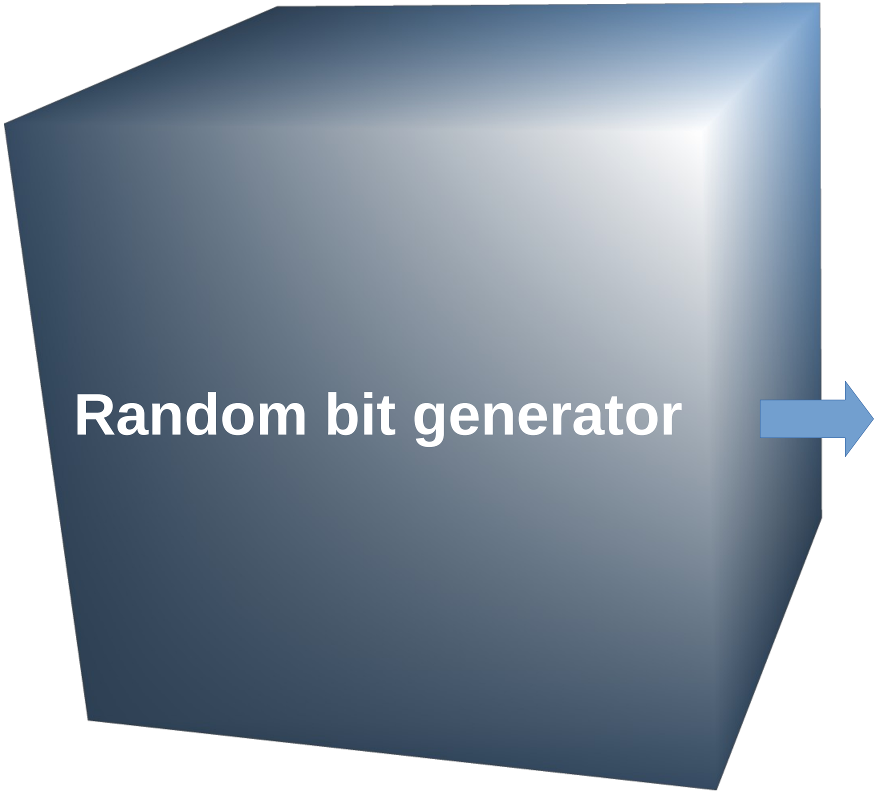


0110100001010111101010111010011101011101

$$H = -p\log_2(p) - (1-p)\log_2(1-p)$$

$$H = ?$$

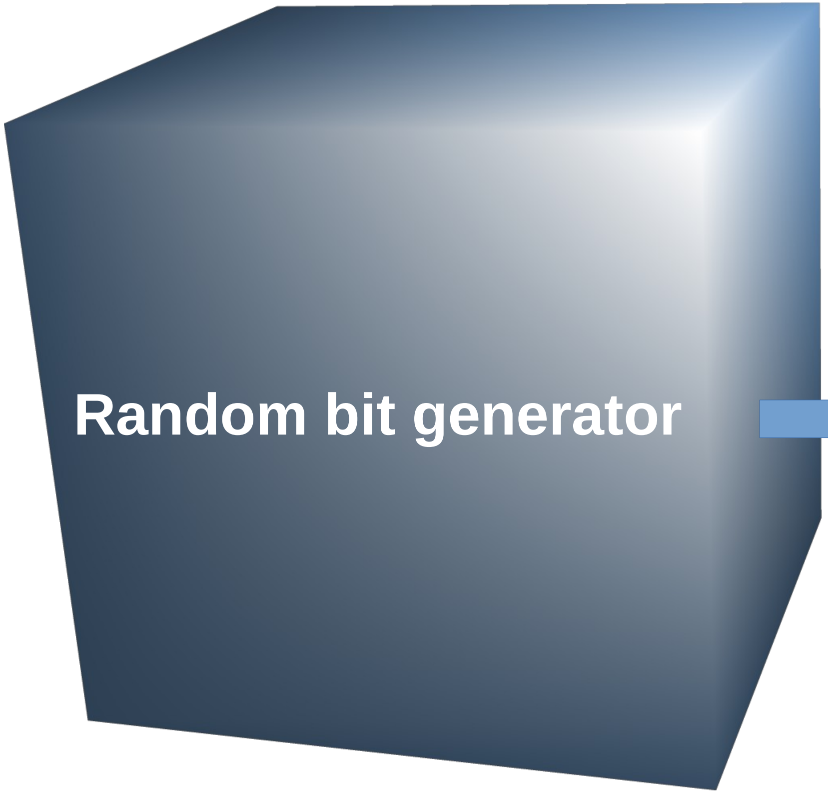
Random data source



0110100001010111101010111010011101011101

Sample of size *N*
with *k* times ‘1‘

Random data source



This allows us to estimate *p* :

$$p \approx \frac{k}{N}$$

$$\underbrace{0110100001010111101010111010011101011101}_{\text{Sample of size } N \text{ with } k \text{ times `1'}}$$

“Naive“ estimation of *H*:

- **Take a sample of size N**
- **Estimate** $p \approx \frac{k}{N}$
- **Plug this estimate into the entropy formula**

$$H = -p\log_2(p) - (1-p)\log_2(1-p)$$

- **Evaluate to get an estimate of *H***

Main message

- **Take a sample of size N**

- **Estimate** $p \approx \frac{k}{N}$

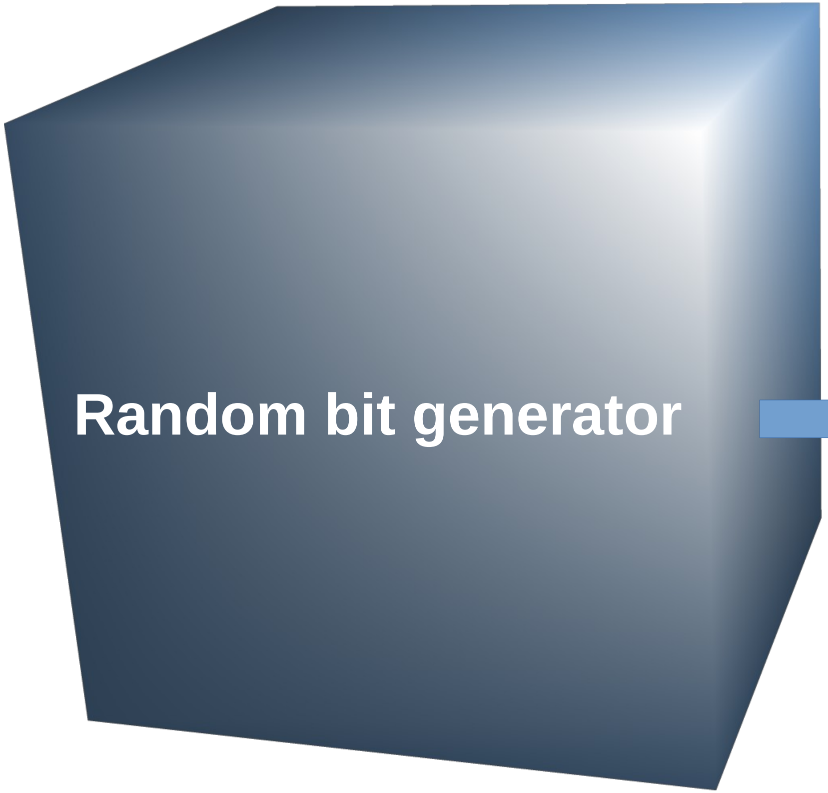
- **Plug this estimate into the entropy formula**

$$H \;=\; -p\log_2(p) - (1-p)\log_2(1-p)$$

- **Evaluate to get an estimate of *H***

Non-linear estimators lead to systematic errors!

Statistical error



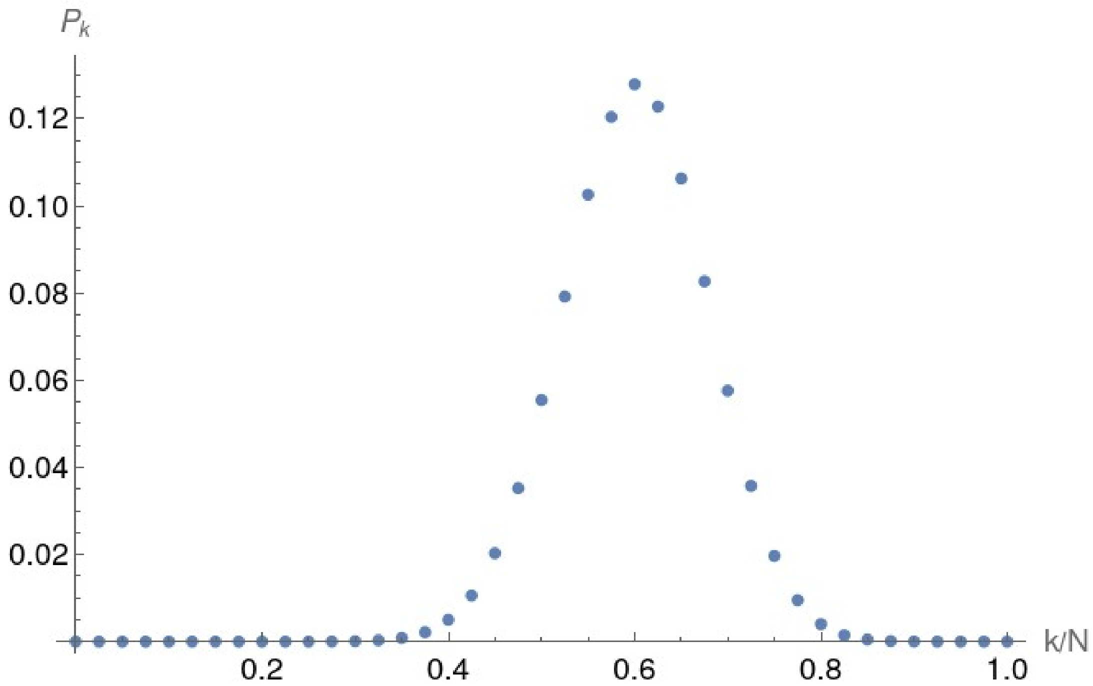
The p-estimates are scattered around the true p:

$$\frac{k}{N} = \frac{23}{40} = 0.575$$

$$\underbrace{0110100001010111101010111010011101011101}$$

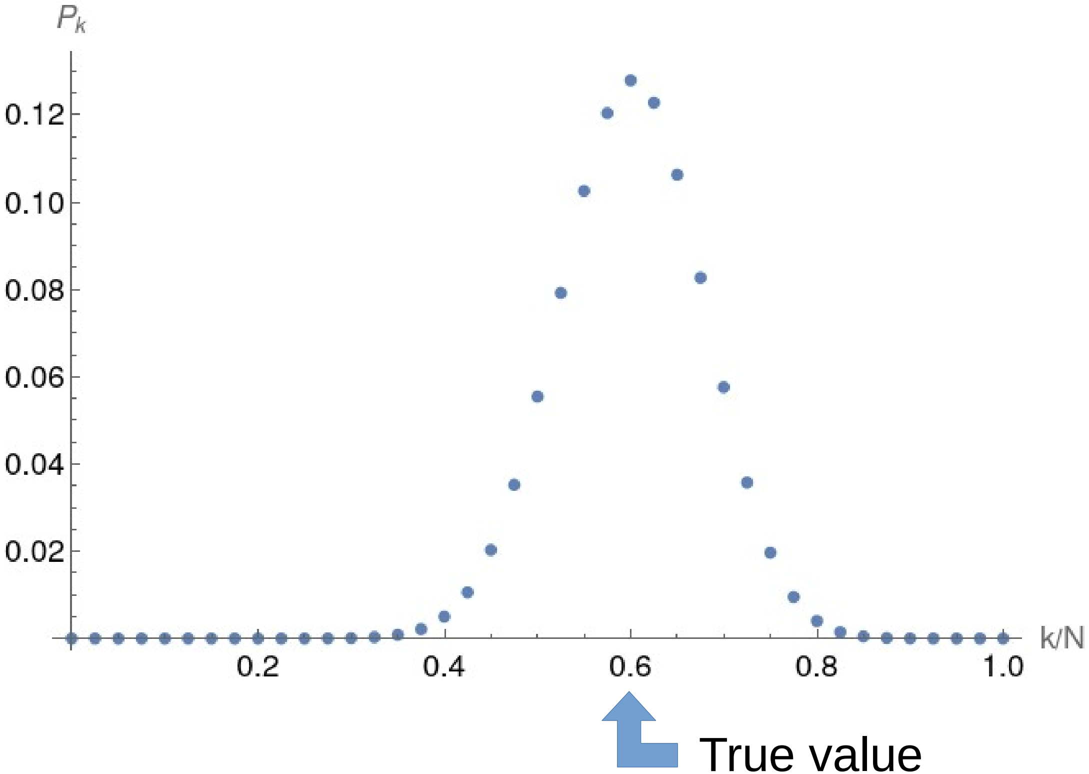
Sample of size *N* with *k* times ‘1‘

Statistical error



True value

Sampling error distribution

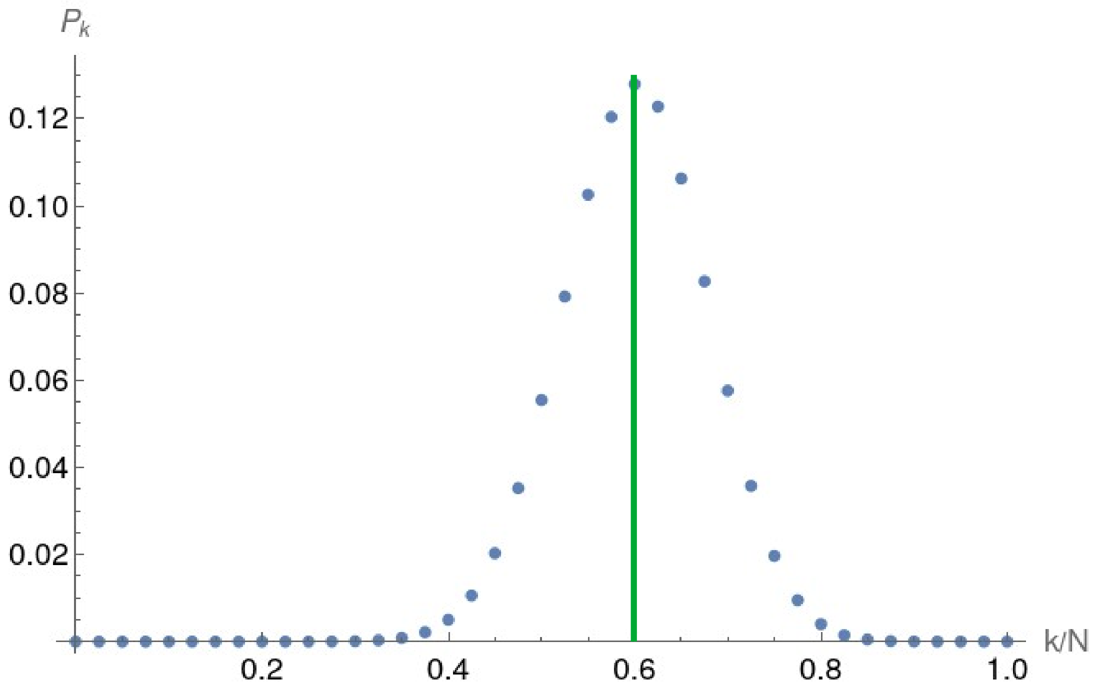


True value

Binomial distribution:

$$P_k = \binom{N}{k} p^k (1-p)^{N-k}$$

Sampling error distribution



$\langle P \rangle$

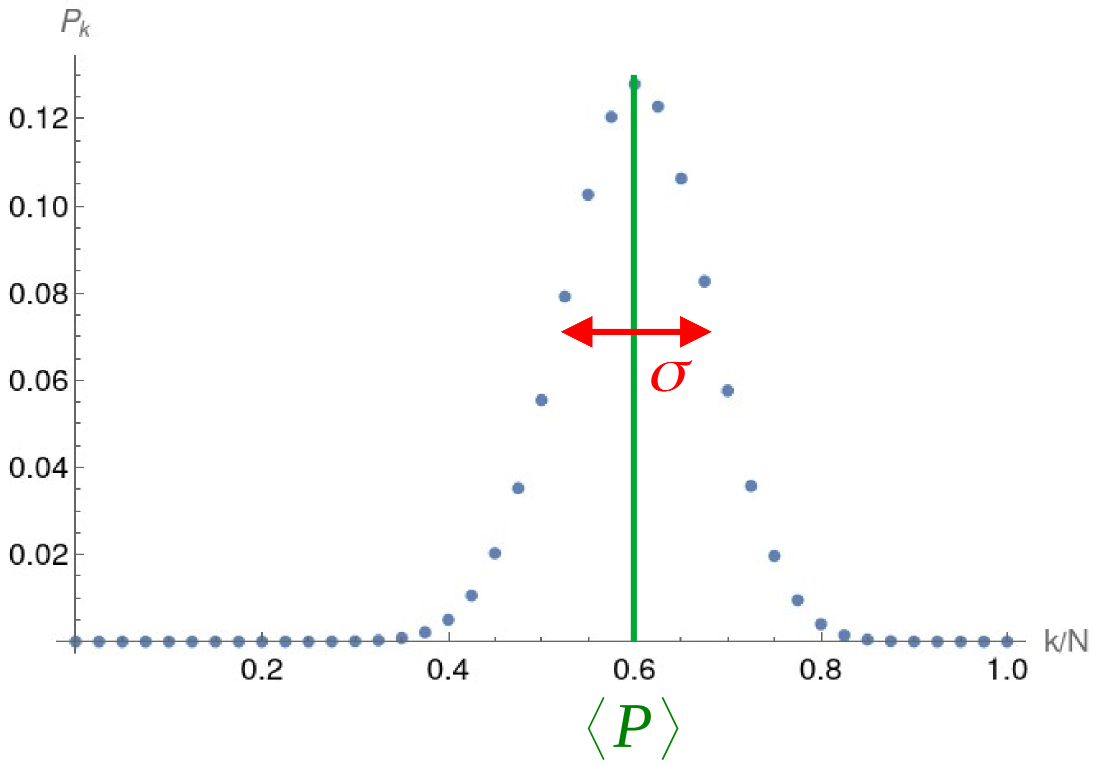
Binomial distribution:

$$P_k = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\langle P \rangle = \sum_{k=1}^{N} P_k \frac{k}{N} = p = 0.6$$

Angled brackets =
Average over many samples

Sampling error distribution



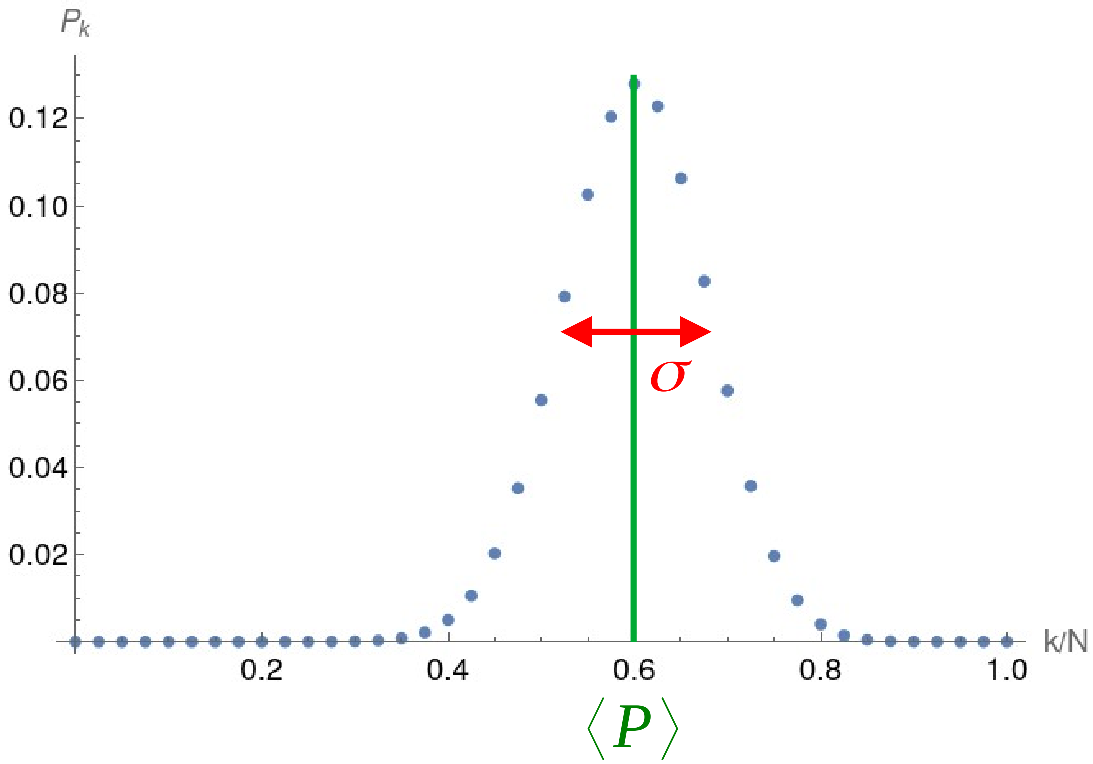
Binomial distribution:

$$P_k = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\langle P \rangle = \sum_{k=1}^{N} P_k \frac{k}{N} = p = 0.6$$

$$\sigma = \sqrt{\frac{p(1-p)}{N}} \approx 0.077$$

Estimation of *p*: No systematic error



Binomial distribution:

$$P_k = \binom{N}{k} p^k (1-p)^{N-k}$$

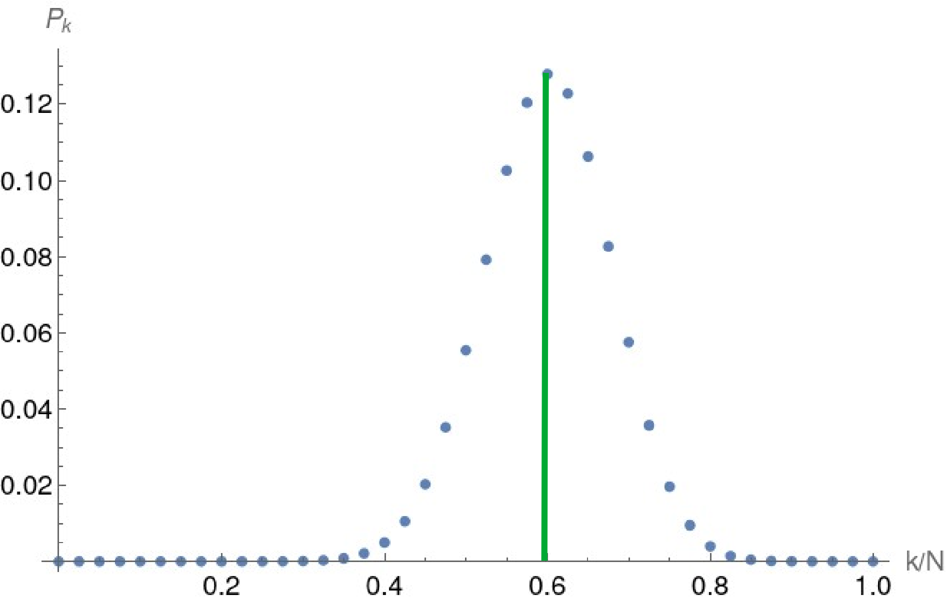
$\langle P \rangle = p$ — No bias
No systematic error

$\sigma > 0$ — Statistical error only

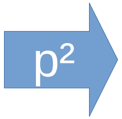
Main message

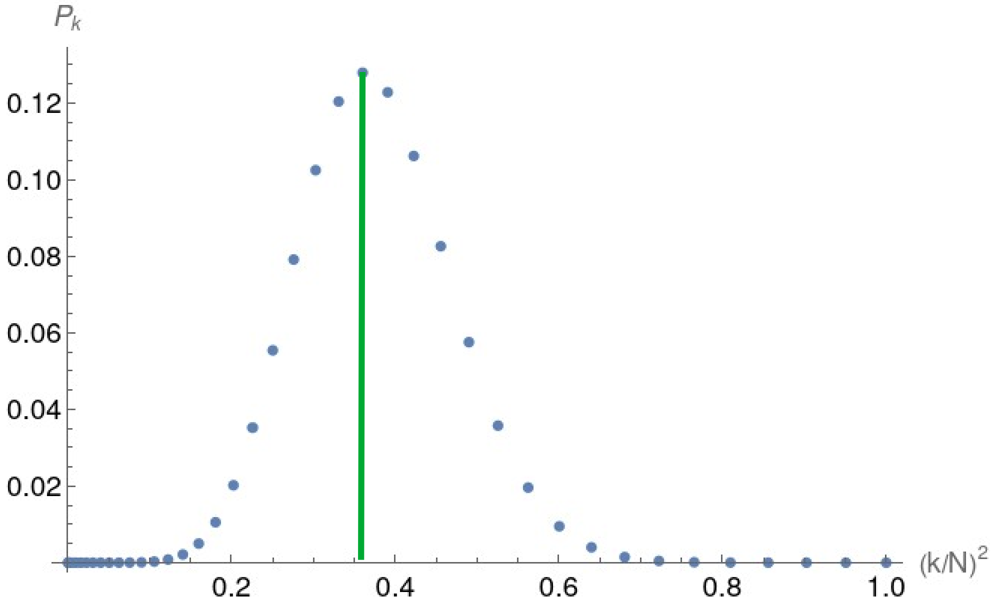
Nonlinear estimators like the entropy lead to <u>systematic</u> errors.

Example of a simple non-linear estimator: *F(p)=p²*



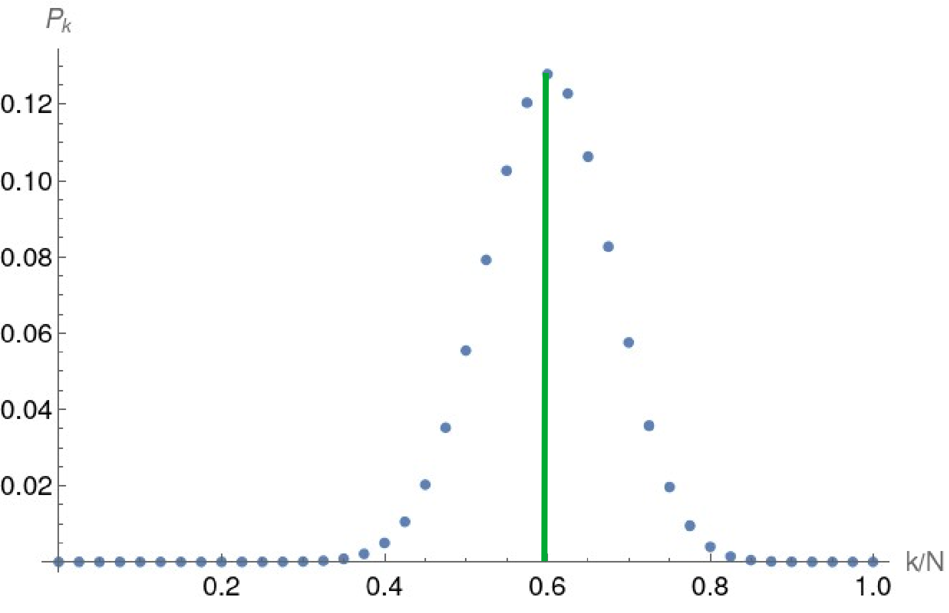
$\langle P \rangle$



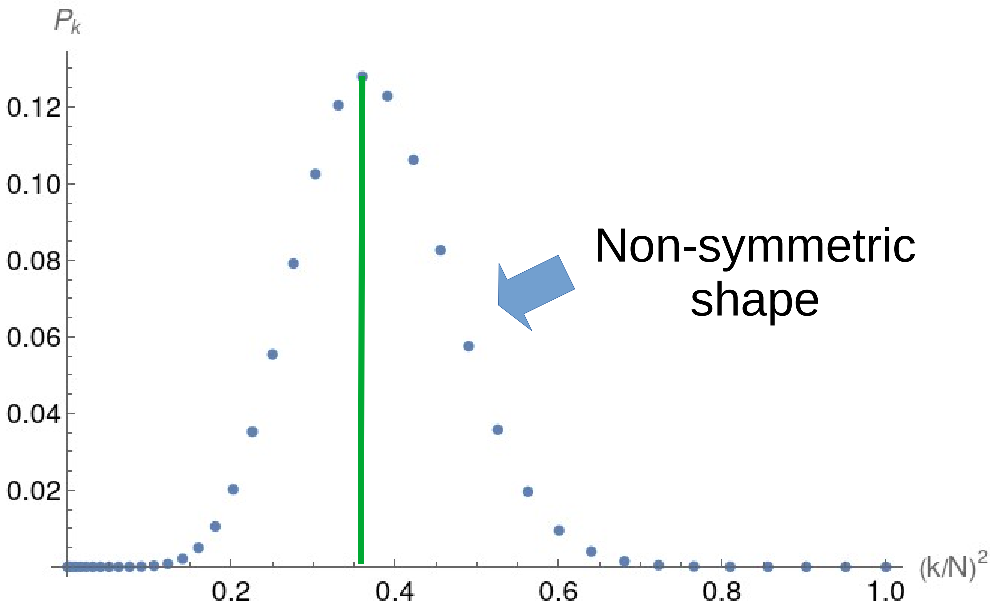


$\langle P \rangle^2$

Example of a simple non-linear estimator: *F(x)=x²*



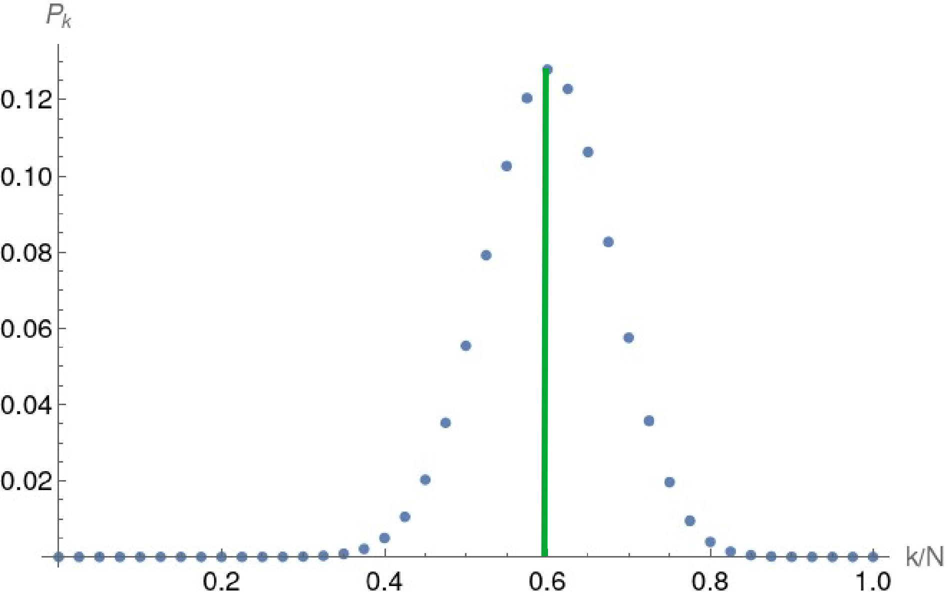
$\langle P \rangle$



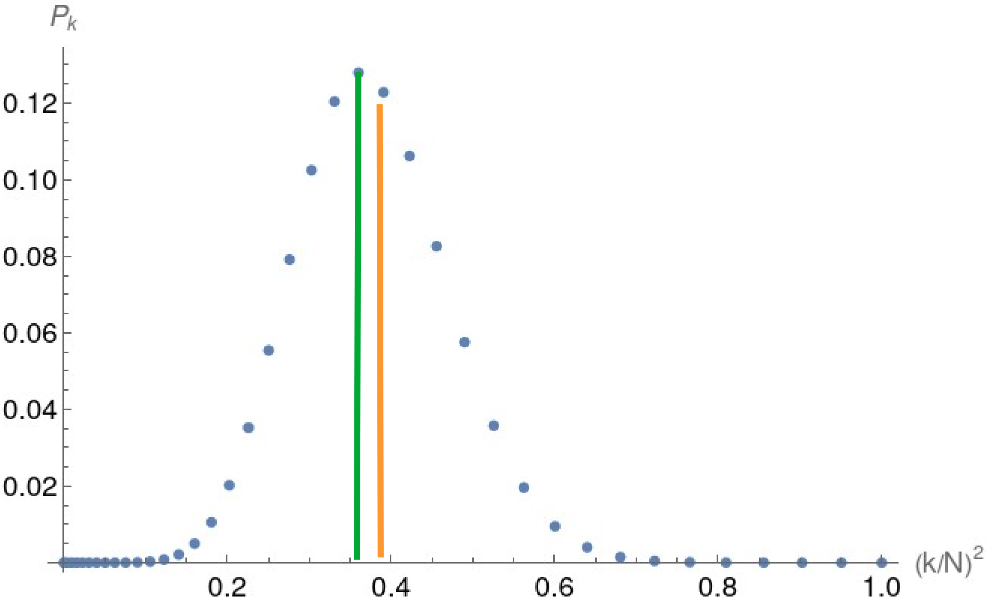
$\langle P \rangle^2$

Non-symmetric shape

Example of a simple non-linear estimator: $F(x)=x^2$



$\langle P \rangle$



$\langle P \rangle^2 \neq \langle P^2 \rangle$

Mathematical origin of this discrepancy:

- **Averaging of a random variable X is a linear operation**

$$\langle X \rangle = \sum_i p_i X_i$$

- **Linear functions *L(x)* commute with averaging**

$$\langle L(X) \rangle = L(\langle X \rangle)$$

- **Nonlinear functions *F(x)* do not commute with averaging**

$$\langle F(X) \rangle \neq F(\langle X \rangle)$$

Systematic errors

How can we detect systematic errors?

Take the smallest sample size!

To see systematic errors, go to small sample sizes

- **Take a sample of size N=1**

- **Estimate** $p \approx \frac{k}{N} = 0,1$

- **Plug the estimate into the entropy formula**

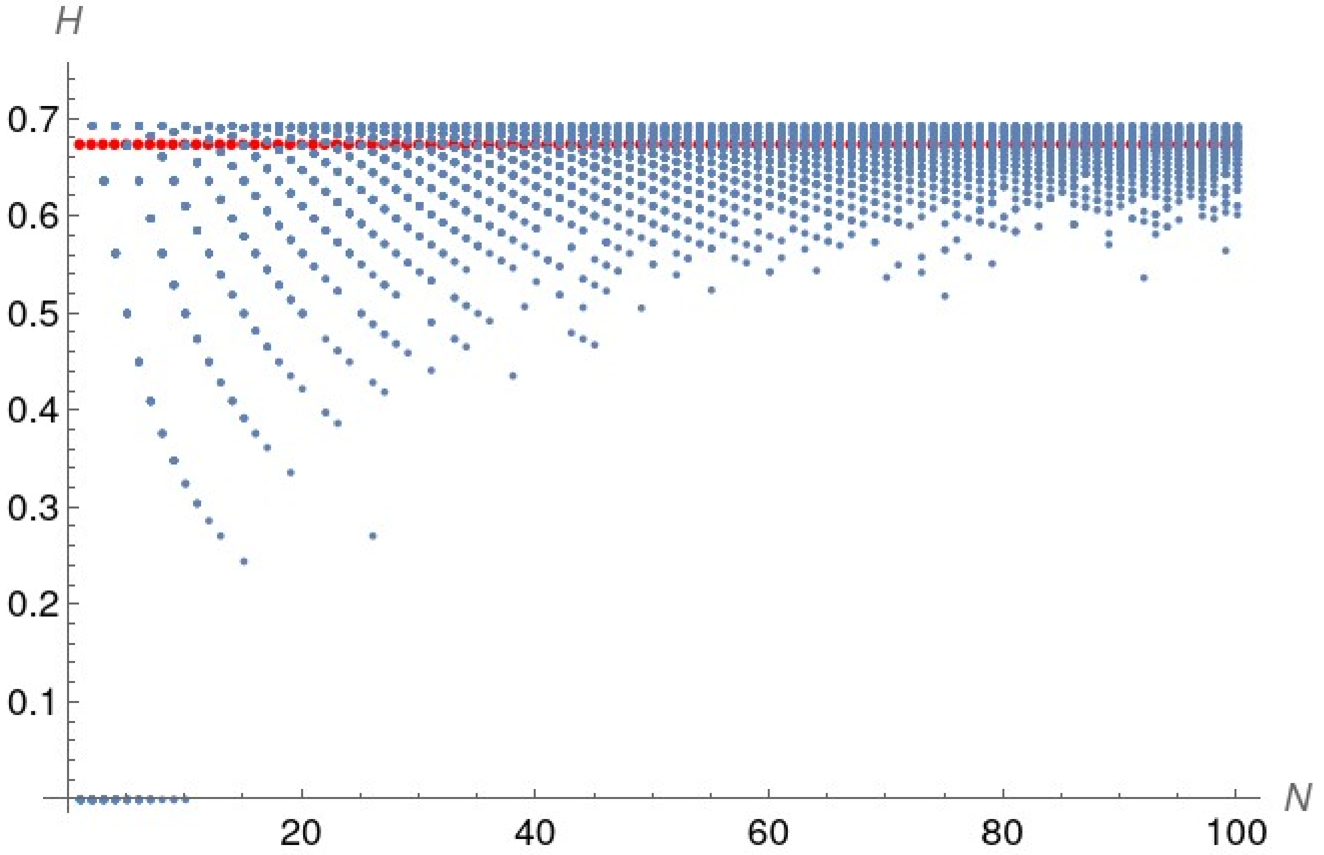
$$H = -p\log_2(p) - (1-p)\log_2(1-p) = 0$$

- **Evaluate to get an estimate of *H***

The entropy is systematically under-estimated.

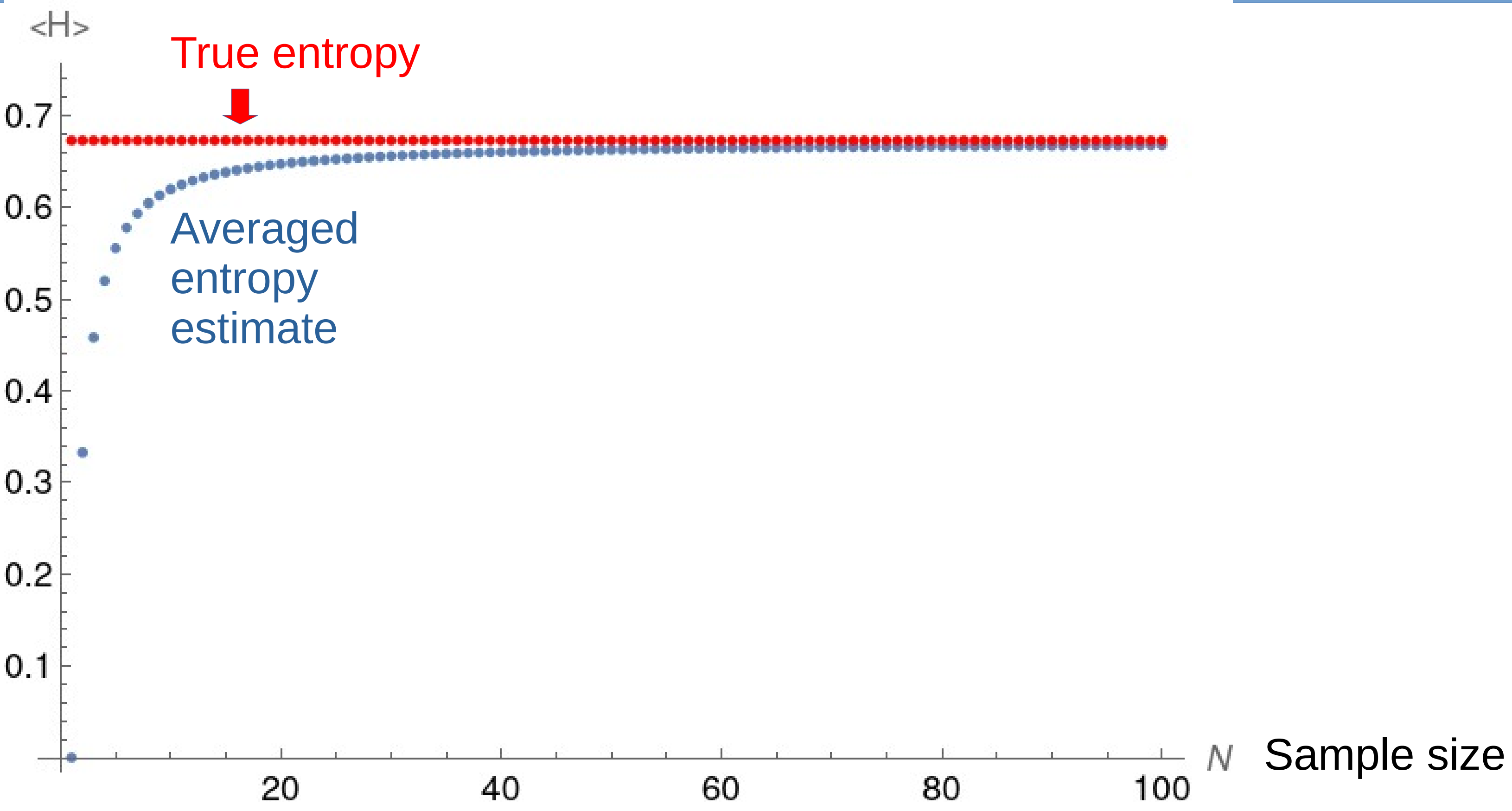
Systematic error underestimating the entropy

Entropy estimate



Sample size

Systematic error underestimating the entropy



The good news:

- Systematic errors are to some extent predictable.
- You can find compensation formulas in the literature.
- The more systematic errors are compensated for, the more this comes at the expense of statistical error.

Compensating systematic errors of entropy estimates

- **Miller (so so simple, no excuse for not using this one)**

  G. Miller (1955), Information theory in Psychology II-B
  ed. H. Quastler, Glencoe, Illinois: Free Press 95

- **Grassberger**

  Finite sample corrections to entropy and dimension estimates,
  Phys. Lett. A 128 (1988) 369

- **Bonachela / H.H. / Munoz**

  Entropy estimates of small data sets,
  J. Phys. A 41 (2008) 202001

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The Miller correction

- **Naive entropy estimate**

$$H_{naive} = -\sum_j \frac{k_j}{N} \ln \frac{k_j}{N}$$

Simply replace p_j by $\frac{k_j}{N}$

The Miller correction

- **Naive entropy estimate**

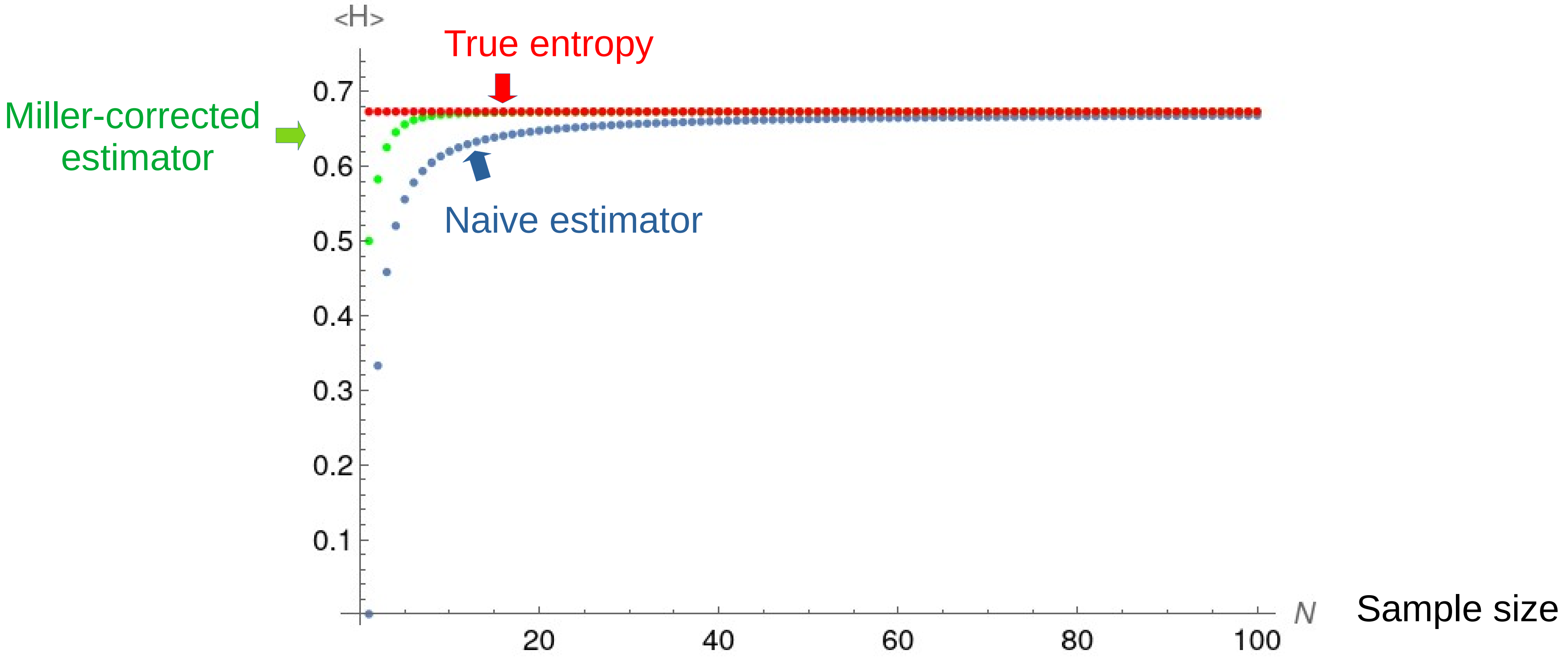
$$H_{naive} = -\sum_j \frac{k_j}{N} \ln \frac{k_j}{N}$$

Simply replace p_j by $\frac{k_j}{N}$

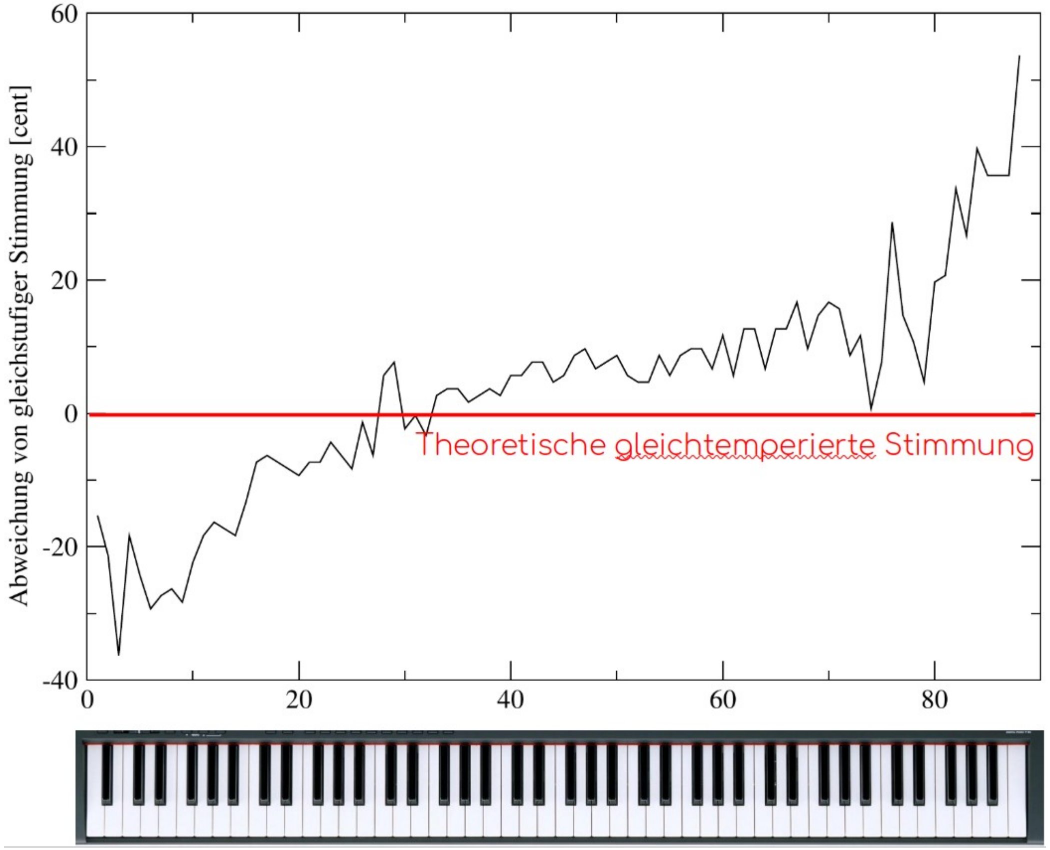
- **Miller-correted entropy estimate**

$$H_{Miller} = -\sum_j \frac{k_j}{N} \ln \frac{k_j}{N} \underbrace{+ \frac{1}{2N}}_{\text{Miller correction}}$$

Miller-corrected entropy estimation

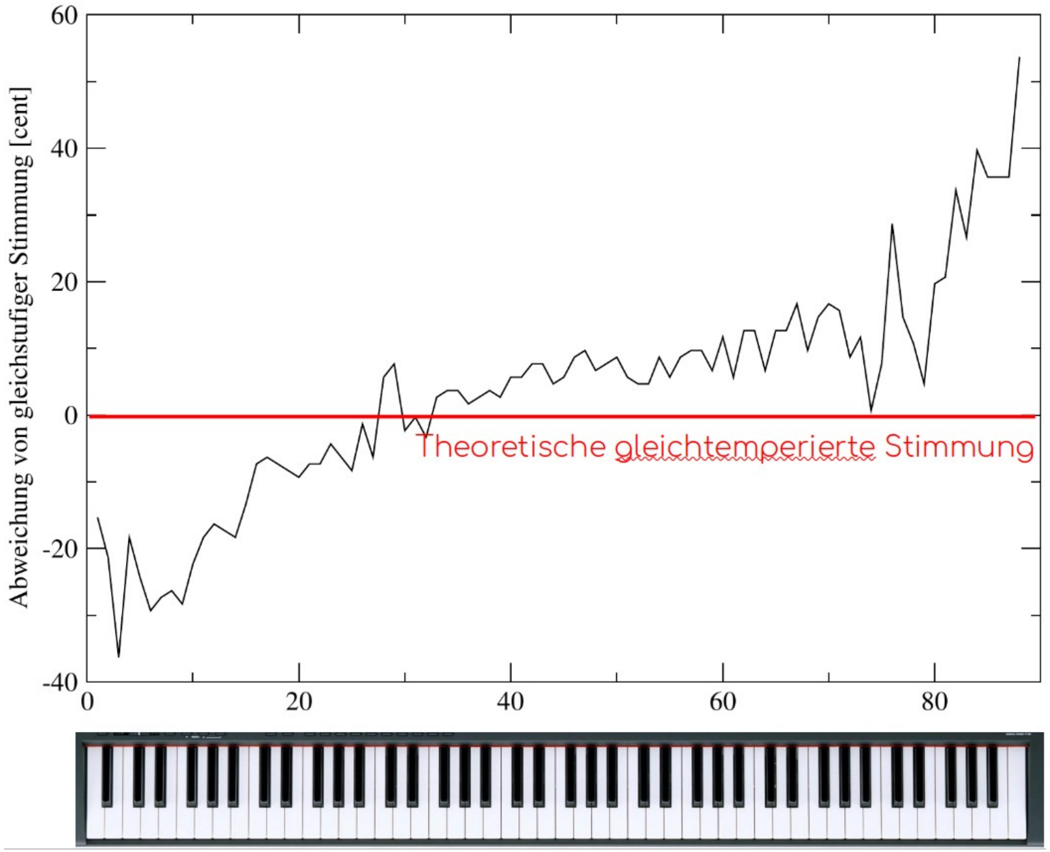


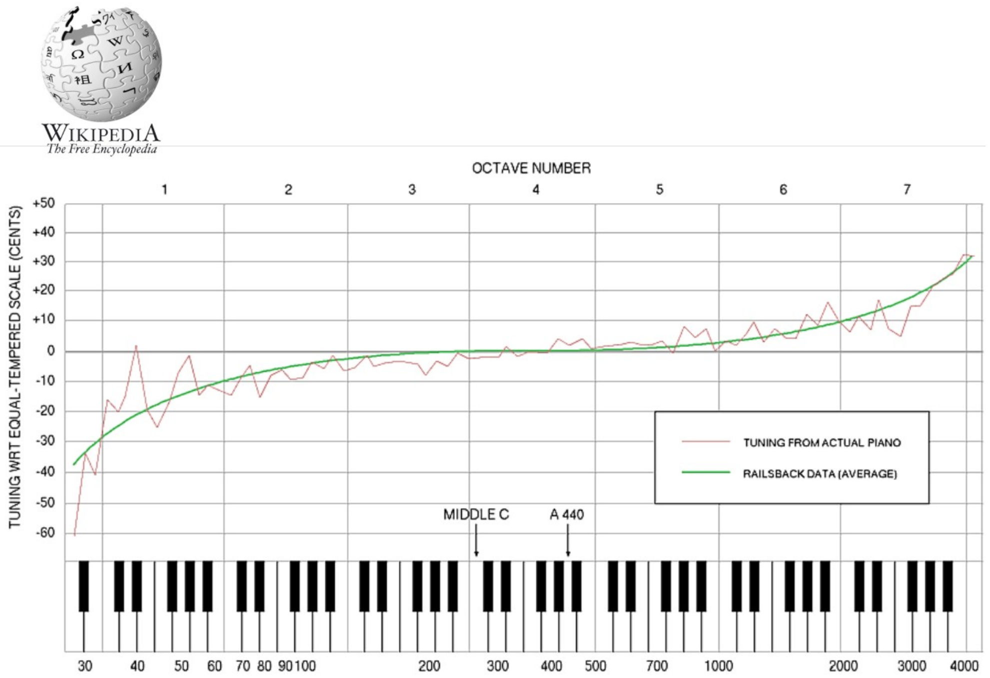
7) Last but not least: A funny story



The frequencies of an aurally tuned piano deviate significantly from the mathematical formula.

Well-tuned pianos are always “out of tune”





Inharmonicity of a realisitic piano string

Ideal string

$$f_n = n \cdot f_1$$

Ideal rod

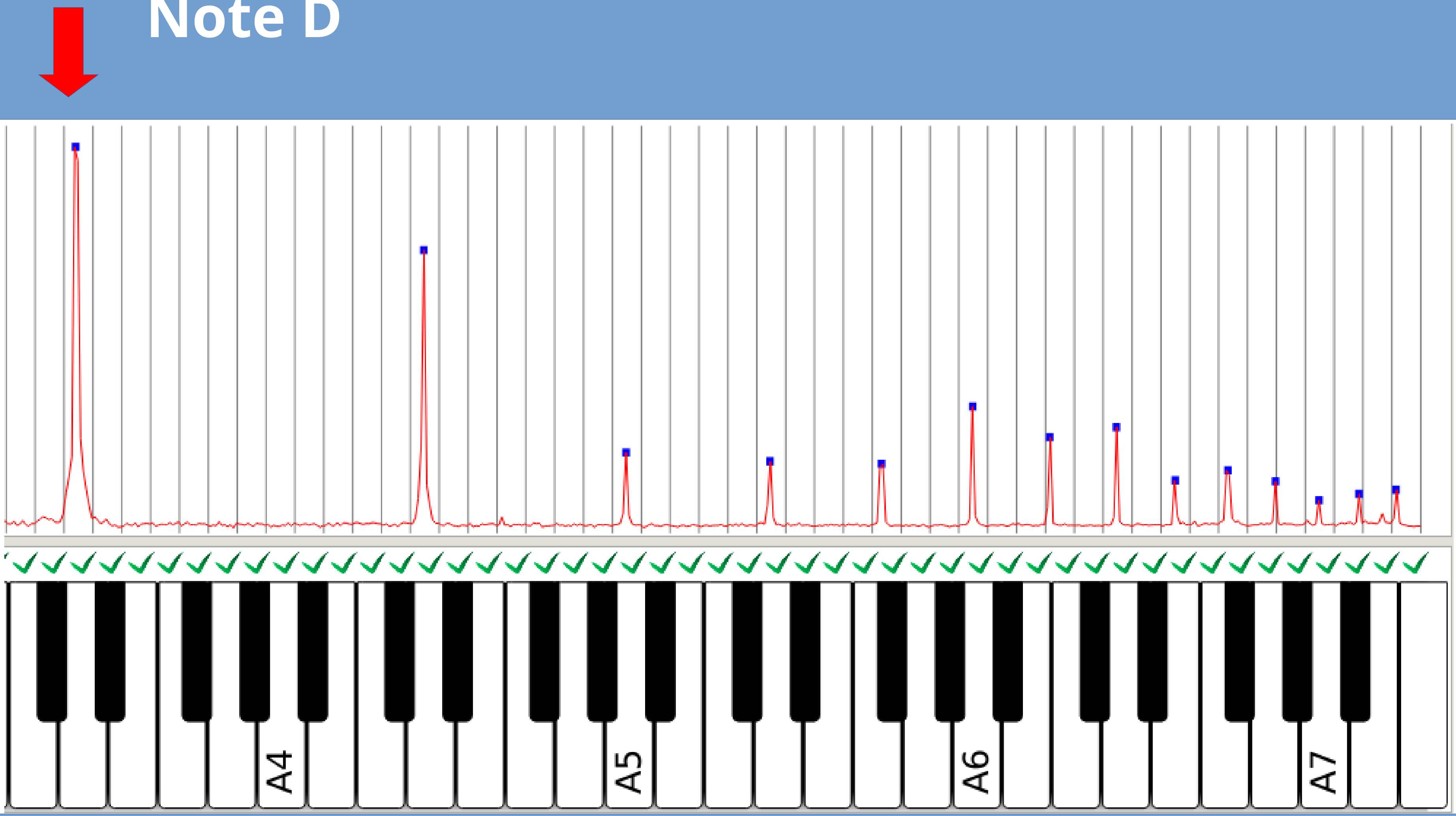
$$f_n = n^2 \cdot f_1$$

Realistic string

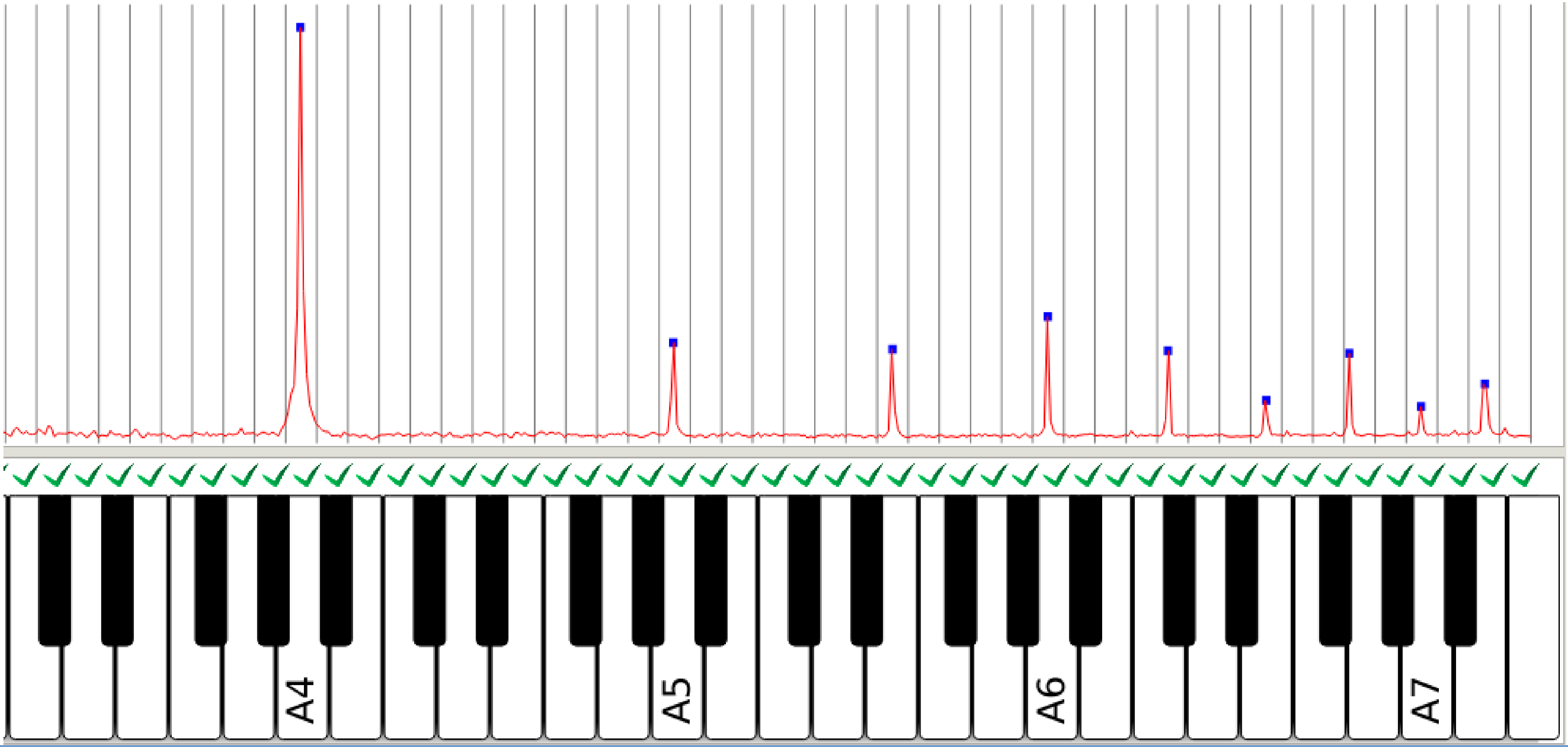
$$f_n = n \cdot f_1 \cdot \sqrt{1 + B n^2}$$

Inharmonicity coefficient

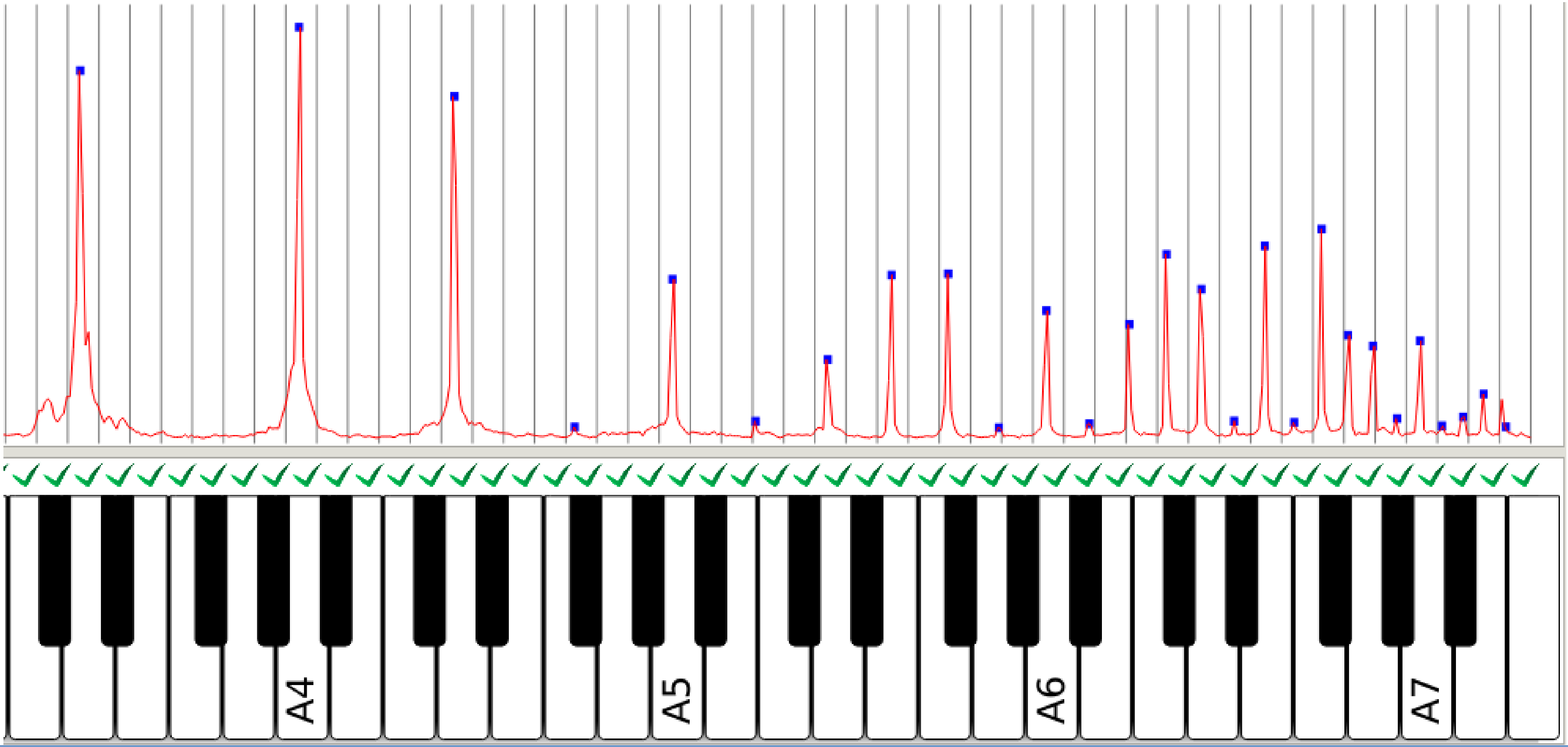
Note D



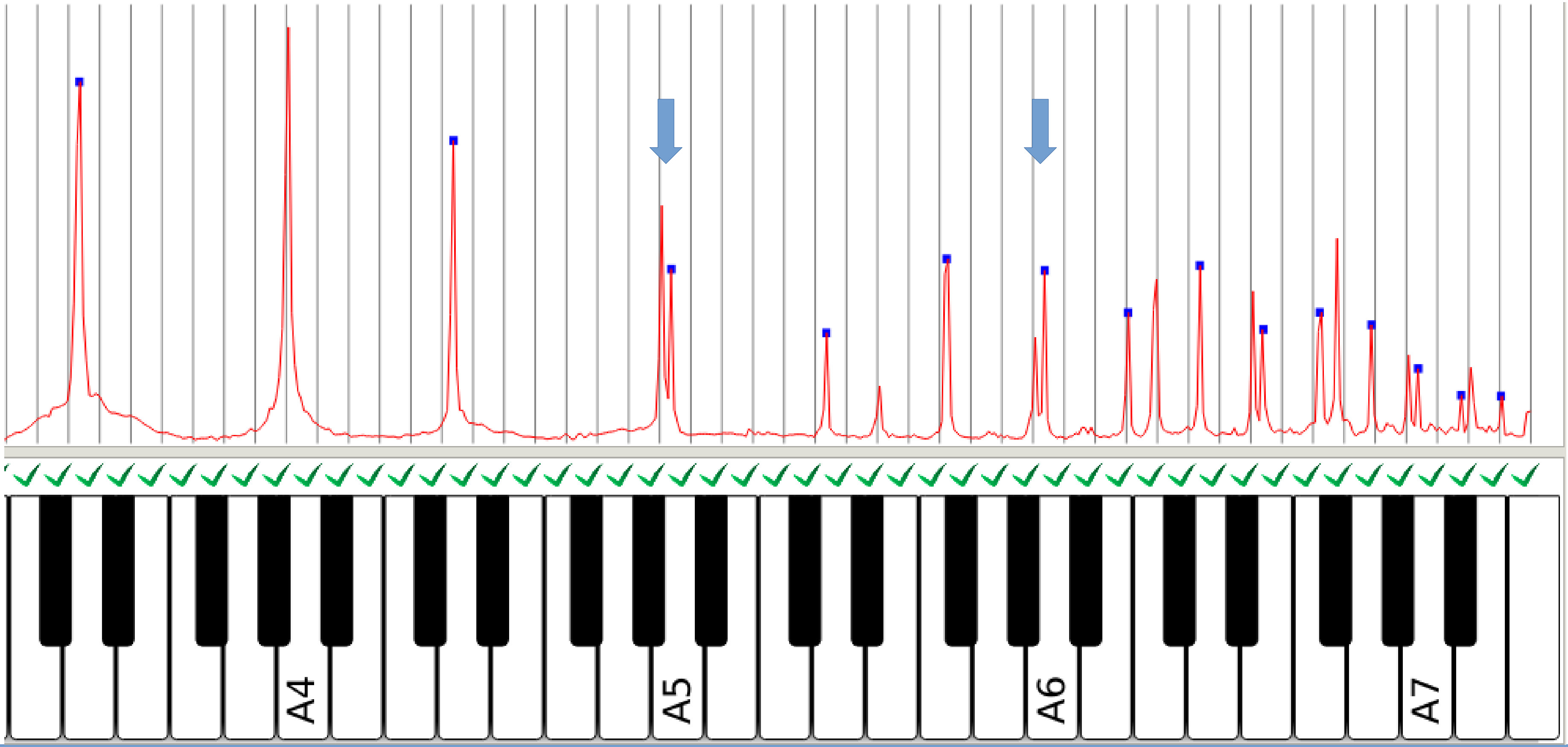
Note A (a fifth higher)



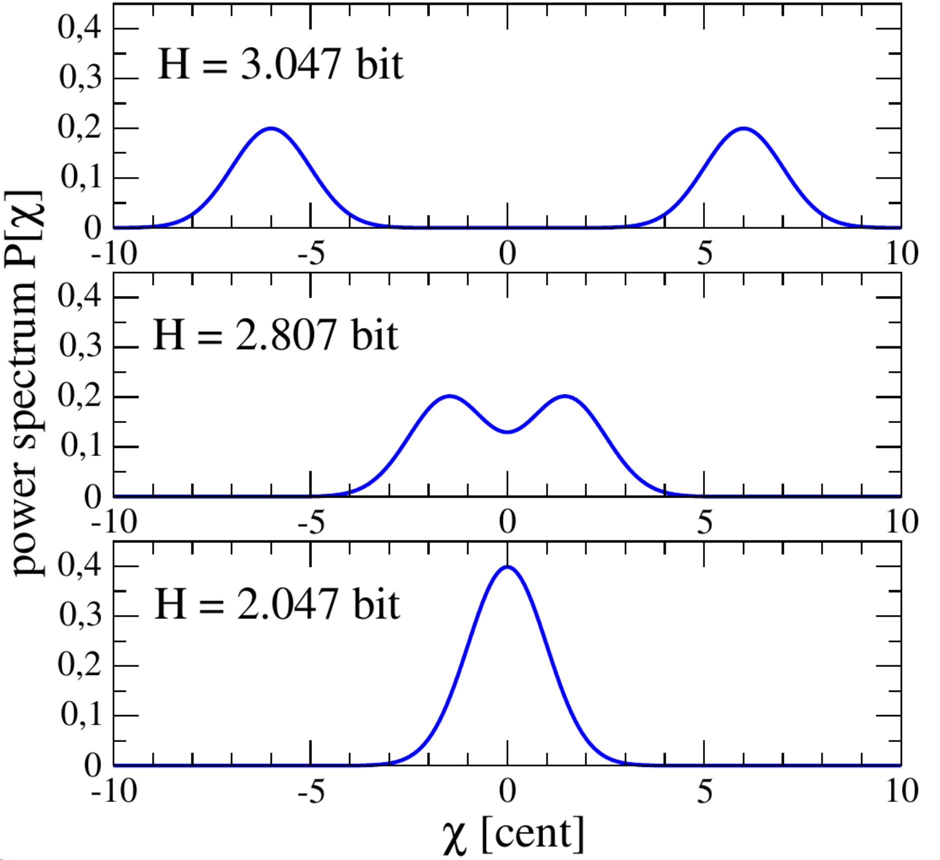
Both together (well tuned)



Both together (out of tune)



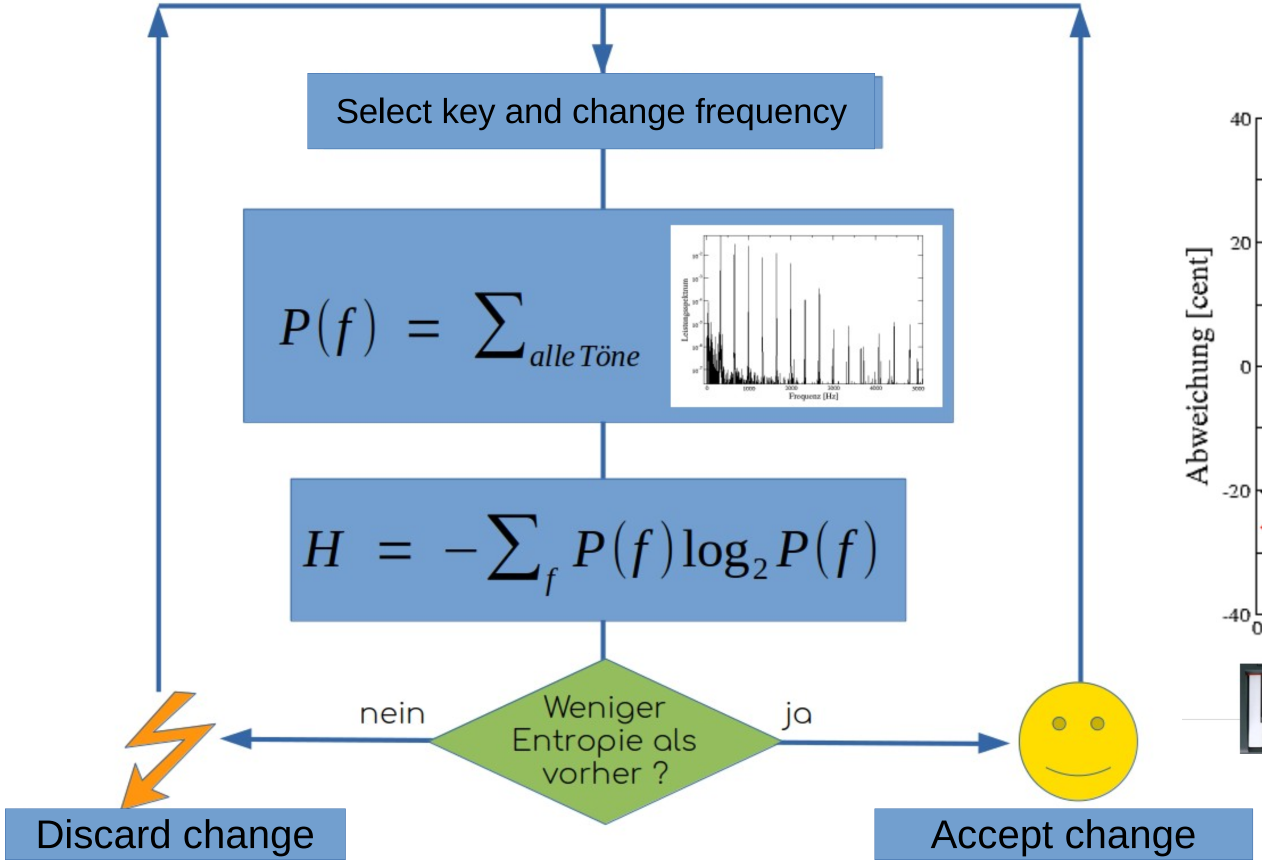
Well tuned = Low entropy of the spectrum

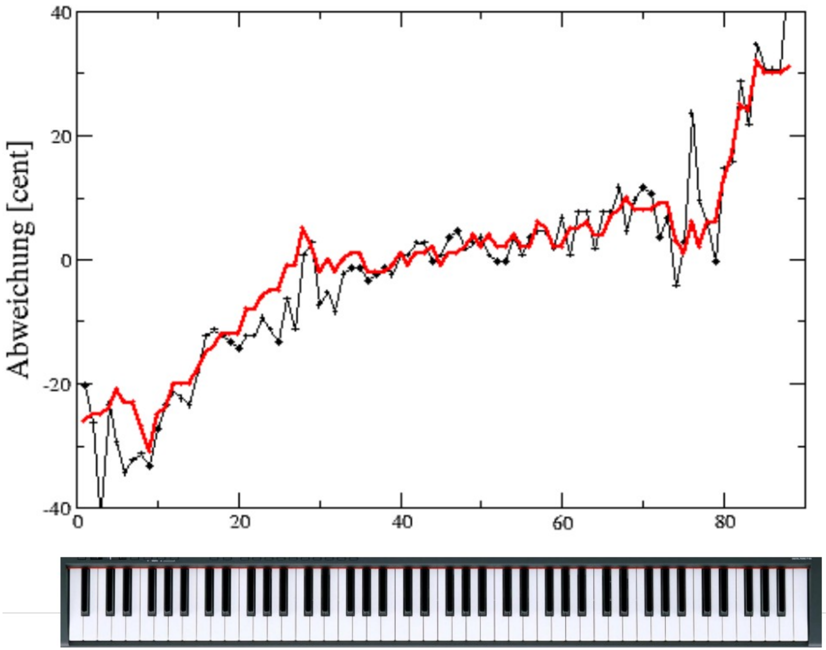


Out of tune = high entropy

Well tuned = low entropy

Entropy-based piano tuning





Wenn Physiker ein Klavier stimmen

Ein Würzburger Physikprofessor veröffentlicht einen Aufsatz. Englischsprachige Medien greifen das Thema mit schmissigen Überschriften auf – und sorgen damit für Unruhe in Musikerkreisen. Die Geschichte eines Missverständnisses.

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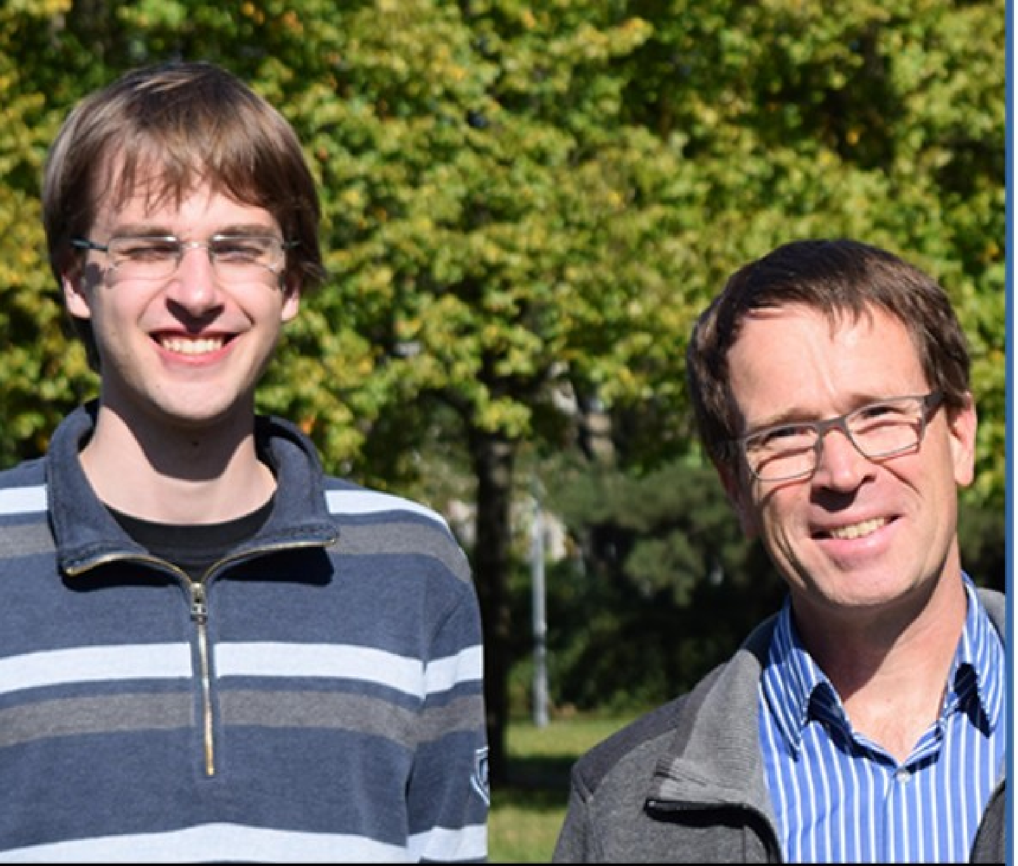
| Columnists

Algorithm captures what sounds most pleasing to the ear - and it could put human instrument tuners out of the job

Arbeitslosigkeit für Klavierstimmer prognostiziert diese Überschrift der Daily Mail. Schuld daran soll der Würzburger Physikprofessor Haye Hinrichsen sein. Screenshot: Robert Emmerich

Haye Hinrichsen ist Professor am Physikalischen Institut der Universität Würzburg. Und er ist Familienvater. Seine Kinder spielen gerne Klavier, und darum schafft die Familie ein solches Instrument an. Ein Klavierstimmer bringt es zum vollendeten Wohlklang. „Wie Physiker eben sind: Bei dieser Aktion kam ich auf die Idee, einfach



Christoph Wick

3.1. Average density matrix

Let us now compute the average density matrix

$$\langle\rho(t)\rangle_{\text{GUE}} = \sum_{i,j=1}^{4} \left\langle e^{-i(E_i-E_j)t}\right\rangle_E \underbrace{\left\langle |\phi_i\rangle\langle\phi_i|\rho(0)|\phi_j\rangle\langle\phi_j|\right\rangle_\alpha}_{=T_{ij}} . \tag{23}$$

Computing the average over the energies

$$\left\langle e^{-i(E_i-E_j)t}\right\rangle_E = \frac{1}{\mathcal{N}}\int_{-\infty}^{+\infty} dE_1\cdots dE_4\, P_{\text{GUE}}(E_1,\ldots,E_4)\, e^{-i(E_i-E_j)t}, \tag{24}$$

where $\mathcal{N} = \int_{-\infty}^{+\infty} dE_1\cdots dE_4\, P_{\text{GUE}}(E_1,\ldots,E_4) = \frac{9\pi}{2A^8}$ is the normalization factor, we obtain the result

$$\left\langle e^{-i(E_i-E_j)t}\right\rangle_E = g(\tau) + (1-g(\tau))\,\delta_{ij} = \begin{cases} 1 & \text{for } i=j \\ g(\tau) & \text{for } i\neq j \end{cases} , \tag{25}$$

where we defined the scaled time $\tau := t/\sqrt{A}$ and the function

$$g(\tau) = \frac{(1152 - 2304\tau^2 + 1104\tau^4 - 256\tau^6 + 25\tau^8 - \tau^{10})}{1152} e^{-\frac{\tau^2}{2}} . \tag{26}$$

Thus Eq. (23) reduces to

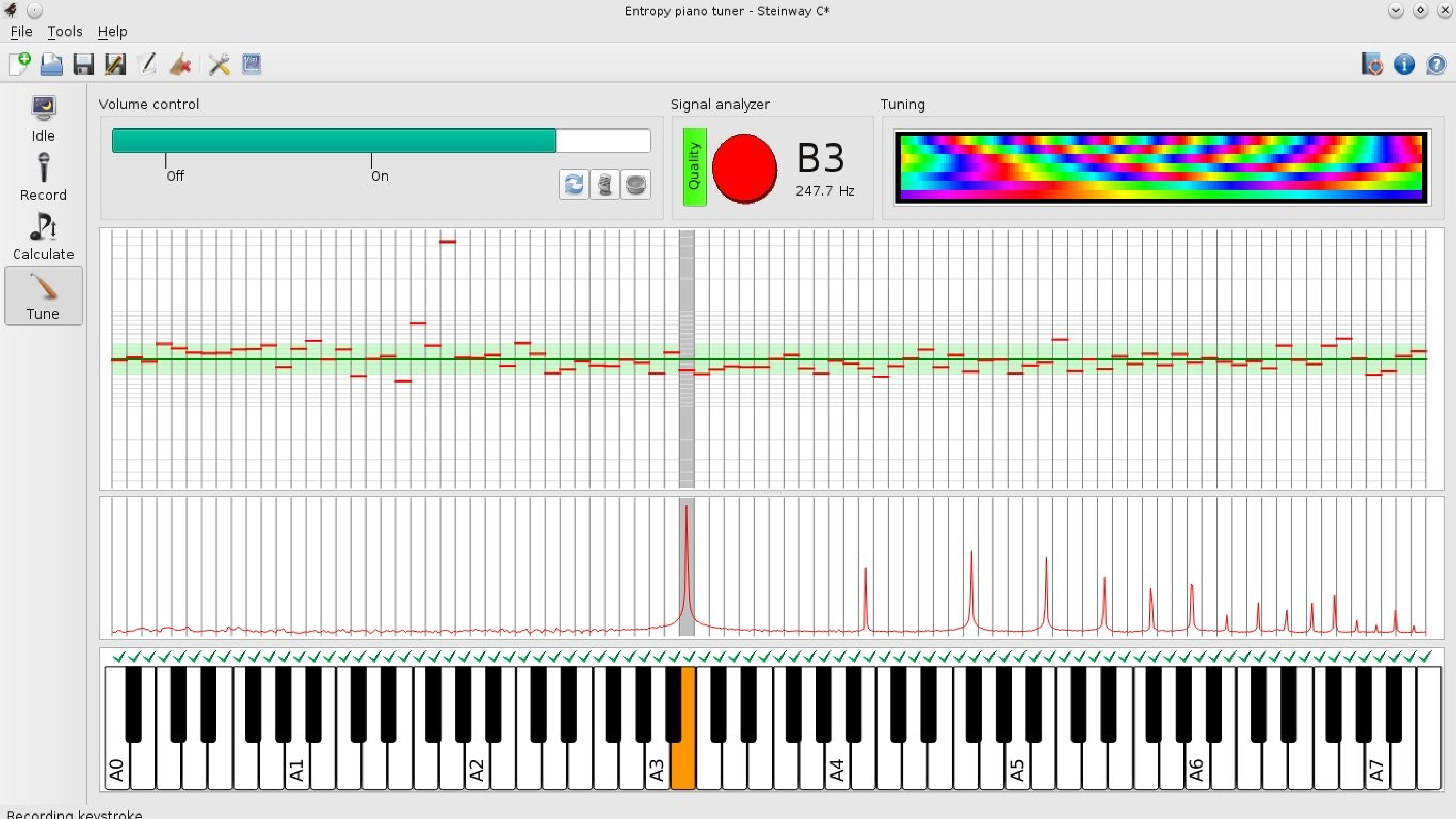
$$\langle\rho(t)\rangle_{\text{GUE}} = g(\tau)\sum_{i,j=1}^{4} T_{ij} + (1-g(\tau))\sum_{i=1}^{4} T_{ii} \tag{27}$$

What remains is to determine the operators

$$T_{ij} = \left\langle U_\alpha^\dagger|i\rangle\langle i|U_\alpha\, \rho(0)\, U_\alpha^\dagger|j\rangle\langle j|U_\alpha\right\rangle_\alpha . \tag{28}$$

Obviously, the first sum in Eq. (27) is given by

$$\sum^{4} T_{ij} = \left\langle U_\alpha^\dagger U_\alpha\, \rho(0)\, U_\alpha^\dagger U_\alpha\right\rangle = \left\langle\rho(0)\right\rangle = \rho(0). \tag{29}$$


Entropy piano tuner - Steinway C*
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Tools
Help
Idle
Record
Calculate
Tune
Volume control
Off
On
Signal analyzer
Quality
B3
247.7 Hz
Tuning
A0
A1
A2
A3
A4
A5
A6
A7
Recording keystroke

Double-blind test at the University of Music, Würburg

Google Play Spiele Apps Filme & Serien Bücher Kinder

Entropy-Piano-Tuner

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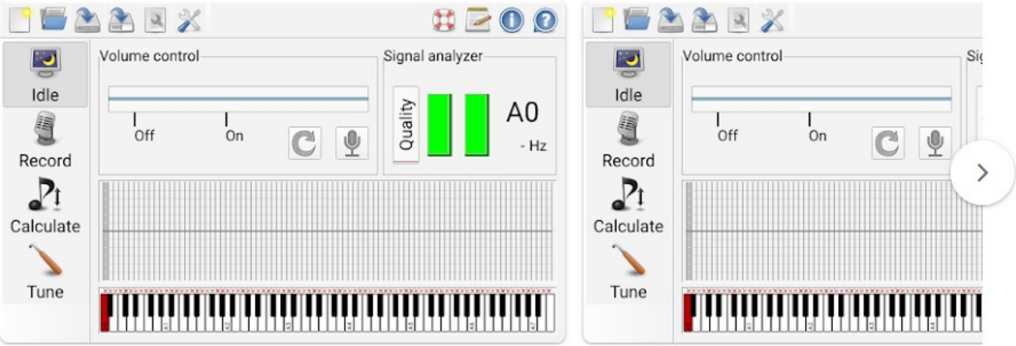

Über diese App →

Stimmen Sie Ihr Klavier oder Ihren Flügel mit Hilfe eines neuen Verfahrens, das auf Entropieminimierung beruht und die individuellen Eigenschaften Ihres Instruments berücksichtigt.

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Summary

1) Clausius – Boltzmann – Gibbs – Shannon – von Neumann
2) Entropy = Information
3) Partial knowledge = Probability
4) Random walk in configuration space → 2nd Law of Thermodynamics
5) Temperature is the price of Energy in terms of Entropy
6) Sampling and the correction of systematic errors
7) Entropy and Piano Tuning

Thank you !