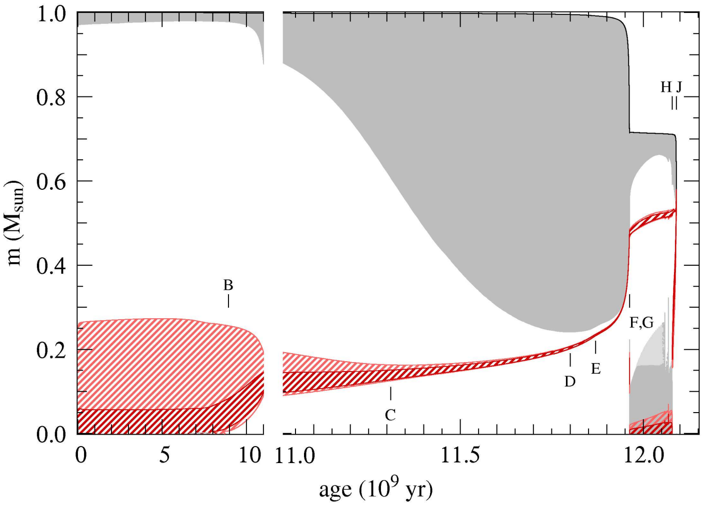
STELLAR STRUCTURE AND EVOLUTION



O.R. Pols

September 2008

Preface

These lecture notes are intended for an advanced astrophysics course on Stellar Structure and Evolution given at Utrecht University (NS-AP434M). Their goal is to provide an overview of the physics of stellar interiors and its application to the theory of stellar structure and evolution, at a level appropriate for a third-year Bachelor student or beginning Master student in astronomy. To a large extent these notes draw on the classical textbook by Kippenhahn & Weigert (1990; see below), but leaving out unnecessary detail while incorporating recent astrophysical insights and up-to-date results. At the same time I have aimed to concentrate on physical insight rather than rigorous derivations, and to present the material in a logical order, following in part the very lucid but somewhat more basic textbook by Prialnik (2000). Finally, I have borrowed some ideas from the textbooks by Hansen, Kawaler & Trimble (2004) and Salaris & Cassisi (2005).

These lecture notes are evolving and I try to keep them up to date. If you find any errors or inconsistencies, I would be grateful if you could notify me by email (`O.R.Pols@uu.nl`).

Onno Pols
Utrecht, August 2008

Literature

- R. Kippenhahn & A. Weigert, *Stellar Structure and Evolution*, 1990, Springer-Verlag, ISBN 3-540-50211-4 (Kippenhahn; K&W)

  Some sections of this book are still used in the lectures, as indicated where appropriate in these notes.

- D. Prialnik, *An Introduction to the Theory of Stellar Structure and Evolution*, 2000, Cambridge University Press, ISBN 0-521-65937-X (Prialnik)

- C.J. Hansen, S.D. Kawaler & V. Trimble, *Stellar Interiors*, 2004, Springer-Verlag, ISBN 0-387-20089-4 (Hansen)

- M. Salaris & S. Cassisi, *Evolution of Stars and Stellar Populations*, 2005, John Wiley & Sons, ISBN 0-470-09220-3 (Salaris)

Physical and astronomical constants

Table 1. Physical constants in cgs units (CODATA 2006).

gravitational constant	G	$6.674\,3 \times 10^{-8}\ \mathrm{cm^3\ g^{-1}\ s^{-2}}$
speed of light in vacuum	c	$2.997\,924\,58 \times 10^{10}\ \mathrm{cm\ s^{-1}}$
Planck constant	h	$6.626\,069 \times 10^{-27}\ \mathrm{erg\ s}$
radiation density constant	a	$7.565\,78 \times 10^{-15}\ \mathrm{erg\ cm^{-3}\ K^{-4}}$
Stefan-Boltzmann constant	$\sigma = \frac{1}{4}ac$	$5.670\,40 \times 10^{-5}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ K^{-4}}$
Boltzmann constant	k	$1.380\,650 \times 10^{-16}\ \mathrm{erg\ K^{-1}}$
electron volt	eV	$1.602\,176\,5 \times 10^{-12}\ \mathrm{erg}$
electron charge	e	$4.803\,26 \times 10^{-10}\ \mathrm{esu}$
	e^2	$1.440\,00 \times 10^{-7}\ \mathrm{eV\ cm}$
electron mass	m_e	$9.109\,382 \times 10^{-28}\ \mathrm{g}$
atomic mass unit	m_u	$1.660\,538\,8 \times 10^{-24}\ \mathrm{g}$
proton mass	m_p	$1.672\,621\,6 \times 10^{-24}\ \mathrm{g}$
neutron mass	m_n	$1.674\,927\,2 \times 10^{-24}\ \mathrm{g}$
α-particle mass	m_α	$6.644\,656\,2 \times 10^{-24}\ \mathrm{g}$

Table 2. Astronomical constants, mostly from the Astronomical Almanac (2008).

Solar mass	$M_\odot$	$1.988\,4 \times 10^{33}\ \mathrm{g}$
	$GM_\odot$	$1.327\,124\,42 \times 10^{26}\ \mathrm{cm^3\ s^{-2}}$
Solar radius	$R_\odot$	$6.957 \times 10^{10}\ \mathrm{cm}$
Solar luminosity	$L_\odot$	$3.842 \times 10^{33}\ \mathrm{erg\ s^{-1}}$
year	yr	$3.155\,76 \times 10^{7}\ \mathrm{s}$
astronomical unit	AU	$1.495\,978\,71 \times 10^{13}\ \mathrm{cm}$
parsec	pc	$3.085\,678 \times 10^{18}\ \mathrm{cm}$

Chapter 1

Introduction

This introductory chapter sets the stage for the course, and briefly repeats some concepts from earlier courses on stellar astrophysics (e.g. the Utrecht first-year course *Introduction to stellar structure and evolution* by F. Verbunt).

1.1 Introduction

The *goal* of this course on stellar evolution can be formulated as follows:

> to understand the structure and evolution of stars, and their observational properties, using known laws of physics

This involves applying and combining 'familiar' physics from many different areas (e.g. thermodyamics, nuclear physics) under extreme circumstances (high T, high ρ), which is part of what makes studying stellar evolution so fascinating.

What exactly do we mean by a 'star'? A useful definition for the purpose of this course is as follows: a star is an object that

- radiates energy from an internal source
- is bound by its own gravity

This definition excludes objects like planets and comets, because they do not comply with the first criterion. In the strictest sense it also excludes brown dwarfs, which are not hot enough for nuclear fusion, although we will briefly discuss these objects. (The second criterion excludes trivial objects that radiate, e.g. glowing coals).

An important implication of this definition is that stars must *evolve* (why?). A star is born out of an interstellar (molecular) gas cloud, lives for a certain amount of time on its internal energy supply, and eventually dies when this supply is exhausted. As we shall see, a second implication of the definition is that stars can have only a limited range of masses, between ~0.1 and ~100 times the mass of the Sun. The *life and death* of stars forms the subject matter of this course. We will only briefly touch on the topic of *star formation*, a complex and much less understood process in which the problems to be solved are mostly very different than in the study of stellar evolution.

1.2 Observational constraints

What do we know from observations?

- photometric measurements yield the apparent brightness of a star, i.e. the energy flux received on Earth, in different wavelength bands. These are usually expressed as magnitudes, e.g. B, V, I, etc. Applying a bolometric correction yields the apparent bolometric flux, $f_{\rm bol}$ (in $\rm erg\,s^{-1}\,cm^{-2}$).

- in some cases, the distance d is known, e.g. from the parallax. The Hipparcos satellite has measured parallaxes with 1 milliarcsec accuracy of more than 10^5 stars. The *luminosity* L of the star (often expressed in $L_\odot = 3.85 \times 10^{33}$ erg/s) then follows from $L = 4\pi d^2 f_{\rm bol}$.

- from spectroscopy, either the overall shape of the spectrum or by detailed spectral-line analysis, the *effective temperature* $T_{\rm eff}$ can be measured. This is defined as the temperature of a blackbody with the same energy flux at the surface of the star, and is a good measure for the temperature of the photosphere. The photometric colours also give a (less accurate) estimate of $T_{\rm eff}$. From the definition of effective temperature it follows that

  $$L = 4\pi R^2 \sigma T_{\rm eff}^4 \tag{1.1}$$

  where R is the radius of the star (often expressed in $R_\odot = 6.96 \times 10^{10}$ cm). Hence if L and $T_{\rm eff}$ are measured, the stellar radius can be inferred.

- detailed spectral-line analysis of the measured spectrum can yield the surface *chemical composition* of the star. Stellar compositions are usually expressed in terms of mass fractions X_i, where i denotes a certain element. This is often simplified to specifying the mass fractions X (of hydrogen), Y (of helium) and Z (of all heavier elements or ‘metals’), which add up to 1.

- for relatively nearby stars, interferometry can yield direct information about stellar radii.

- for direct information about stellar *masses* one needs binary stars with measured radial velocity variations (spectroscopic binaries). In particular for so called double-lined eclipsing binaries, in which the spectral lines of both stars are seen and that furthermore show eclipses, it is possible to accurately measure (with 1–2 % accuracy is some cases):

  - masses from fitting the radial-velocity curve
  - radii from fitting the eclipse lightcurve

  Together with a photometric or, better, spectroscopic determination of $T_{\rm eff}$ also the luminosity of such binaries can be measured with high accuracy, independent of the distance.

All observed properties are surface properties. Therefore we need a *theory of stellar structure* to derive the internal properties of a star. However, some direct windows on the inside of a star exist:

- *neutrinos*, which escape from the interior without interaction. So far, the Sun is the only (non-exploding) star from which neutrinos have been detected.

- *oscillations*, i.e. stellar seismology. Many stars oscillate, and their frequency spectrum contains information about the speed of sound waves inside the star, and therefore about the interior density and temperature profiles. This technique has provided accurate constraints on detailed structure models for the Sun, and is now also being applied to other stars.

In addition, the timespan of any observations is much smaller than a stellar lifetime: observations are like snapshots in the life of a star. The observed properties of an individual star contain no (direct) information about its evolution. The diversity of stellar properties (radii, luminosities, surface abundances) does, however, depend on how stars evolve, as well as on intrinsic properties (mass, initial composition). Properties that are common to a large number of stars must correspond to long-lived evolution phases, and vice versa. By studying samples of stars statistically we can infer the (relative) lifetimes of certain phases, which provides another important constraint on the theory of stellar evolution.

Furthermore, observations of samples of stars reveal certain correlations between stellar properties that the theory of stellar evolution must explain. Most important are relations between luminosity and effective temperature, as revealed by the *Hertzsprung-Russell diagram*, and relations between mass, luminosity and radius.

1.2.1 The Hertzsprung-Russell diagram

The Hertzsprung-Russell diagram (HRD) is an important tool to test the theory of stellar evolution. Fig. 1.1 shows the colour-magnitude diagram (CMD) of stars in the vicinity of the Sun, for which the Hipparcos satellite has measured accurate distances. This is an example of a *volume-limited* sample of stars. In this observers' HRD, the absolute visual magnitude M_V is used as a measure of the luminosity and a colour index ($B - V$ or $V - I$) as a measure for the effective temperature. It is left as an exercise to identify various types of stars and evolution phases in this HRD, such as the main sequence, red giants, the horizontal branch, white dwarfs, etc.

Star clusters provide an even cleaner test of stellar evolution. The stars in a cluster formed within a short period of time (a few Myr) out of the same molecular cloud and therefore share the same age

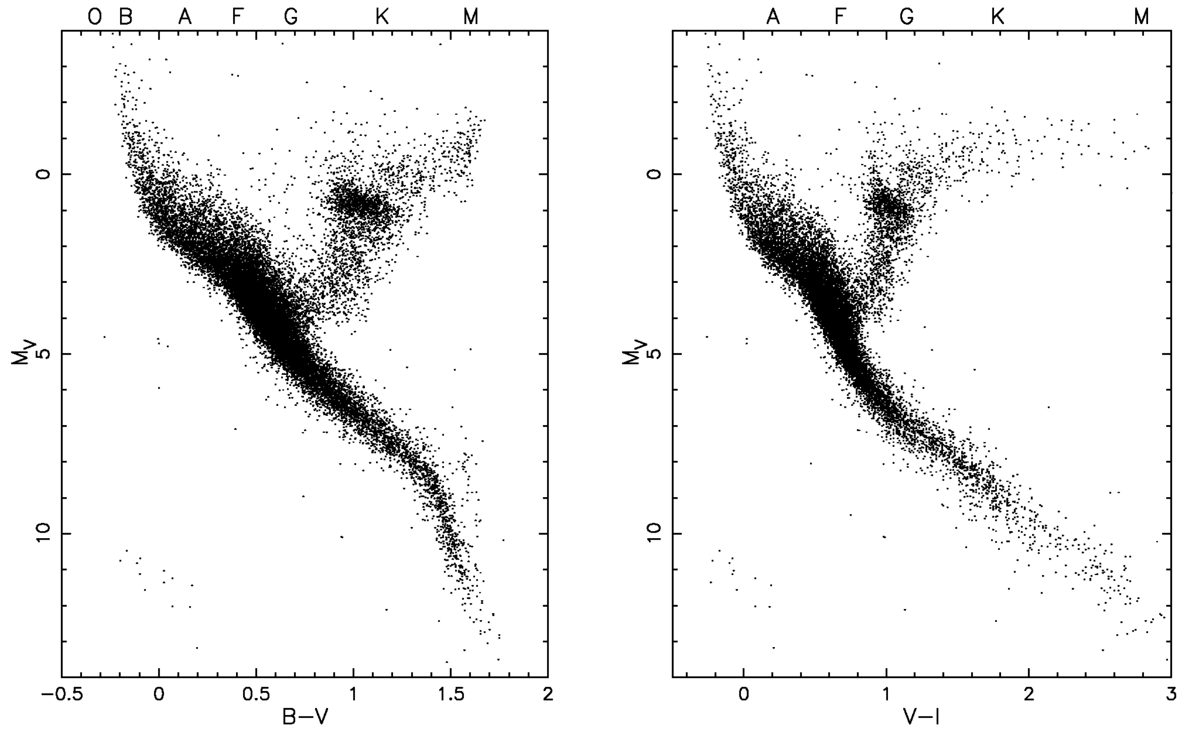


Figure 1.1. H-R diagram of solar neighbourhood. Source: Hipparcos, stars with d measured to $< 10\,\%$ accuracy.

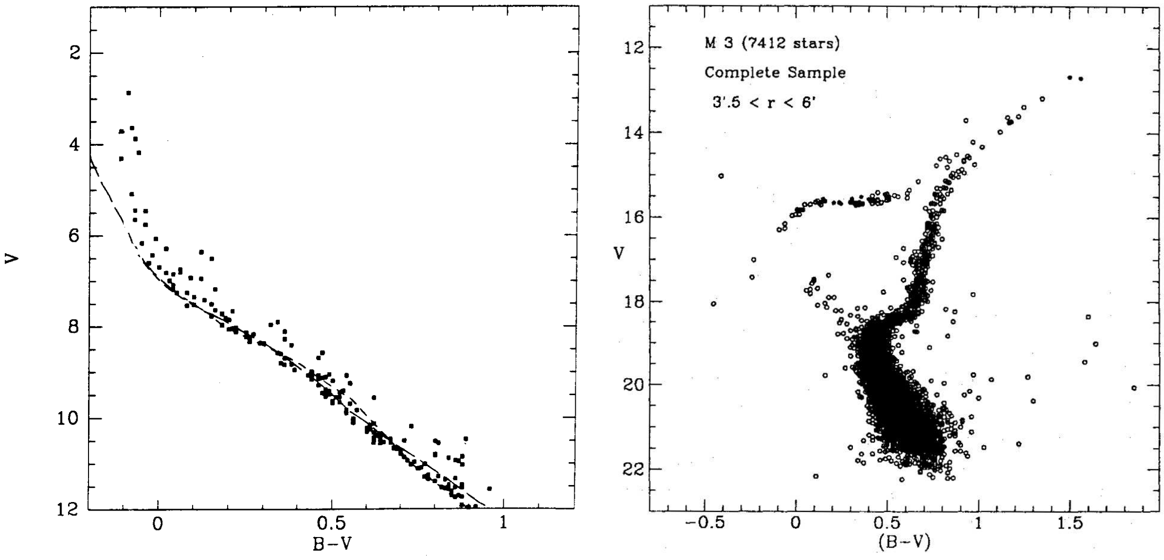


Figure 1.2. Colour-magnitude diagrams of a young open cluster, M45 (the Pleiades, left panel), and a globular cluster, M3 (right panel).

and (initial) chemical composition. Therefore, only the mass varies from star to star. A few examples of cluster CMDs are given in Fig. 1.2, for a young open cluster (the Pleiades) and an old globular cluster (M3). As the cluster age increases, the most luminous main-sequence stars disappear and a prominent red giant branch and horizontal branch appear. To explain the morphology of cluster HRDs at different ages is one of the goals of studying stellar evolution.

1.2.2 The mass-luminosity and mass-radius relations

For stars with measured masses, radii and luminosities (i.e. binary stars) we can plot these quantities against each other. This is done in Fig. 1.3 for the components of double-lined eclipsing binaries for which M, R and L are all measured with $\lesssim 2\,\%$ accuracy. These quantities are clearly correlated, and especially the relation between mass and luminosity is very tight. Most of the stars in Fig. 1.3 are long-lived main-sequence stars; the spread in radii for masses between 1 and $2\,M_\odot$ results from the fact that several more evolved stars in this mass range also satisfy the 2 % accuracy criterion. The observed relations can be approximated reasonably well by power laws:

$$L \propto M^{3.8} \qquad \text{and} \qquad R \propto M^{0.7}. \tag{1.2}$$

Again, the theory of stellar evolution must explain the existence and slopes of these relations.

1.3 Stellar populations

Stars in the Galaxy are divided into different populations:

- Population I: stars in the galactic disk, in spiral arms and in (relatively young) open clusters. These stars have ages $\lesssim 10^9$ yr and are relatively metal-rich ($Z \sim 0.5 - 1\,Z_\odot$)

- Population II: stars in the galactic halo and in globular clusters, with ages $\sim 10^{10}$ yr. These stars are observed to be metal-poor ($Z \sim 0.01 - 0.1\,Z_\odot$).

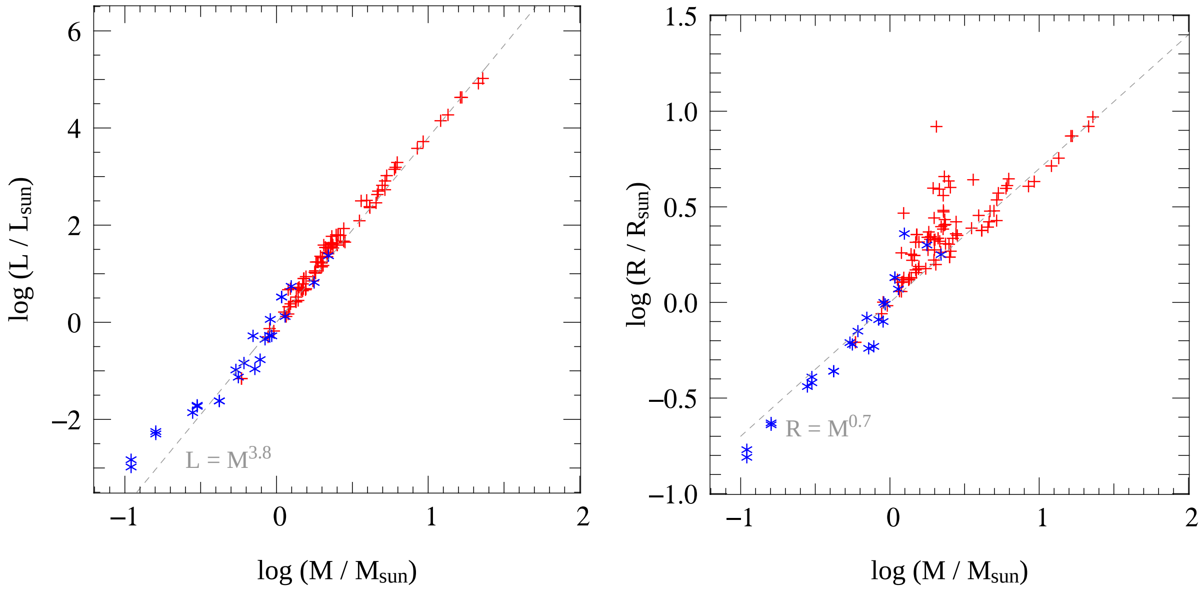


Figure 1.3. Mass-luminosity (left) and mass-radius (right) relations for components of double-lined eclipsing binaries with accurately measured M, R and L.

An intermediate population (with intermediate ages and metallicities) is also seen in the disk of the Galaxy. Together they provide evidence for the *chemical evolution* of the Galaxy: the abundance of heavy elements (Z) apparently increases with time. This is the result of chemical enrichment by subsequent stellar generations.

The study of chemical evolution has led to the hypothesis of a 'Population III' consisting of the first generation of stars formed after the Big Bang, containing only hydrogen and helium and no heavier elements ('metal-free', $Z = 0$). No metal-free stars have ever been observed, probably due to the fact that they were massive and had short lifetimes and quickly enriched the Universe with metals. However, a quest for finding their remnants has turned up many very metal-poor stars in the halo, with the current record-holder having an iron abundance $X_{\mathrm{Fe}} = 4 \times 10^{-6}$ solar.

1.4 Basic assumptions

We wish to build a theory of stellar evolution to explain the observational constraints highlighted above. In order to do so we must make some basic assumptions:

- stars are considerd to be *isolated* in space, so that their structure and evolution depend only on *intrinsic* properties (mass and composition). For most single stars in the Galaxy this condition is satisfied to a high degree (compare for instance the radius of the Sun with the distance to its nearest neighbour Proxima Centauri, see exercises). However, for stars in dense clusters, or in binary systems, the evolution can be influenced by interaction with neighbouring stars. In this course we will mostly ignore these complicating effects (many of which are treated in the Master course on *Binary Stars*).

- stars are formed with a *homogeneous composition*, a reasonable assumption since the molecular clouds out of which they form are well-mixed. We will often assume a so-called 'quasi-solar' composition ($X = 0.70$, $Y = 0.28$ and $Z = 0.02$), even though recent determinations of solar abundances have revised the solar metallicity down to $Z = 0.015$. In practice there is relatively little variation in composition from star to star, so that the initial mass is the most important parameter that determines the evolution of a star. The composition, in particular the metallicity Z, is of secondary influence but can have importtant effects especially in very metal-poor stars (see § 1.3).

- *spherical symmetry*, which is promoted by self-gravity and is a good approximation for most stars. Deviations from spherical symmetry can arise if non-central forces become important relative to gravity, in particular rotation and magnetic fields. Although many stars are observed to have magnetic field, the field strength (even in highly magnetized neutron stars) is always negligible compared to gravity. Rotation can be more important, although for the majority of stars (e.g. the Sun) the forces involved are small compared to gravity. However, some rapidly rotating stars are seen (by means of interferometry) to be substantially flattened.

1.5 Aims and overview of the course

In the remainder of this course we will:

- understand the global properties of stars: energetics and timescales
- study the microphysics relevant for stars: the equation of state, nuclear reactions, energy transport and opacity
- derive the equations necessary to model the internal structure of stars
- examine (quantitatively) the properties of simplified stellar models
- survey (mostly qualitatively) how stars of different masses evolve, and the endpoints of stellar evolution (white dwarfs, neutron stars)
- discuss a few ongoing research areas in stellar evolution

Suggestion for further reading

The contents of this introductory chapter are also largely covered by Chapter 1 of PRIALNIK, which provides nice reading. (Be aware, however, that the lower end of the mass-luminosity relation shown in Fig. 1.6 is wrong, it has a slope that is too large!)

Chapter 2

Mechanical and thermal equilibrium

In this chapter we apply the physical principles of mass conservation, momentum conservation and energy conservation to derive three of the fundamental stellar structure equations. We shall see that stars are generally in a state of almost complete *mechanical equilibrium*, which allows us to derive and apply the important *virial theorem*. We consider the basic stellar timescales and see that most (but not all) stars are also in a state of energy balance called *thermal equilibrium*.

2.1 Coordinate systems and the mass distribution

The assumption of spherical symmetry implies that all interior physical quantities (such as density ρ, pressure P, temperature T, etc) depend only on one radial coordinate. The obvious coordinate to use in a Eulerian coordinate system is the radius of a spherical shell, r ($\in 0 \ldots R$).

In an evolving star, all quantities also depend on time t but this is not explicitly noted in the following: a derivative d/dr (or d/dm) should be taken to mean the partial derivative with respect to the space coordinate, at constant time.

The principle of mass conservation (assuming a steady state, i.e. ignoring a time-varying mass flow) yields the mass dm of a spherical shell of thickness dr at radius r (see Fig. 2.1) as

$$dm = \rho\, dV = \rho\, 4\pi r^2 dr \quad \Rightarrow \quad \frac{dm}{dr} = 4\pi r^2 \rho. \tag{2.1}$$

Note that $\rho = \rho(r)$ is not known a priori, and must follow from other conditions and equations. The differential form of the above equation is therefore the first fundamental equation of stellar structure. Integration yields the mass $m(r)$ inside a spherical shell of radius r:

$$m(r) = \int_0^r 4\pi r'^2 \rho\, dr'. \tag{2.2}$$

Since $m(r)$ increases monotonically outward, we can also use $m(r)$ as our radial coordinate, instead of r. This *mass coordinate*, often denoted as m_r or simply m, is a Lagrangian coordinate that moves with the mass shells:

$$m := m_r = \int_0^r 4\pi r'^2 \rho\, dr' \qquad (m \in 0 \ldots M) \tag{2.3}$$

It is often more convenient to use a Lagrangian coordinate instead of a Eulerian coordinate. The mass coordinate is defined on a fixed interval, $m \in 0 \ldots M$, as long as the star does not lose mass. On the other hand r depends on the time-varying stellar radius R. Furthermore the mass coordinate follows the mass elements in the star, which simplifies many of the time derivatives that appear in the stellar

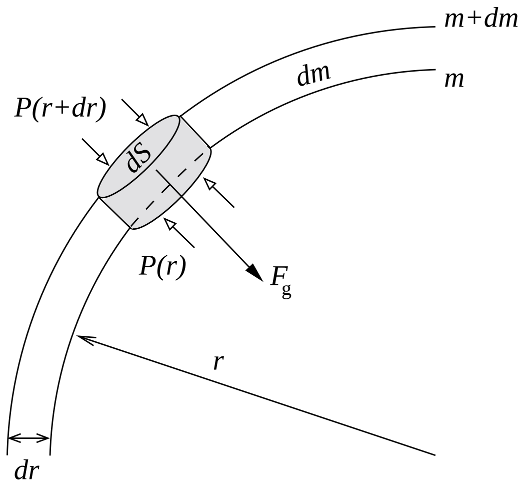


Figure 2.1. Mass shell inside a spherically symmetric star, at radius r and with thickness dr. The mass of the shell is $dm = 4\pi r^2 \rho\, dr$. The pressure and the gravitational force acting on a cylindrical mass element are also indicated.

evolution equations (e.g. equations for the composition). We can thus write all quantities as functions of m, i.e. $r = r(m)$, $\rho = \rho(m)$, $P = P(m)$, etc.

Using the coordinate transformation $r \to m$, i.e.

$$\frac{d}{dm} = \frac{d}{dr} \cdot \frac{dr}{dm} \tag{2.4}$$

the first equation of stellar structure becomes in terms of the coordinate m:

$$\boxed{\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}} \tag{2.5}$$

2.1.1 The gravitational field

Recall that a star is a self-gravitating body of gas, which implies that gravity is the driving force behind stellar evolution. In the general, non-spherical case, the gravitational acceleration $\vec{g}$ can be written as the gradient of the gravitational potential, $\vec{g} = -\vec{\nabla}\Phi$, where Φ is the solution of the Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho.$$

Inside a spherically symmetric body, this reduces to $g := |\vec{g}| = d\Phi/dr$. The gravitational acceleration at radius r and equivalent mass coordinate m is then given by

$$g = \frac{Gm}{r^2}. \tag{2.6}$$

Spherical shells outside r apply no net force, so that g only depends on the mass distribution inside the shell at radius r. Note that g is the magnitude of the vector $\vec{g}$ which points inward (toward smaller r or m).

2.2 The equation of motion and hydrostatic equilibrium

We next consider conservation of momentum inside a star, i.e. Newton’s second law of mechanics. The net acceleration on a gas element is determined by the sum of all forces acting on it. In addition to the gravitational force considered above, forces result from the pressure exerted by the gas surrounding the element. Due to spherical symmetry, the pressure forces acting horizontally (perpendicular to the radial direction) balance each other and only the pressure forces acting along the radial direction need to be considered. By assumption we ignore other forces that might act inside a star (Sect. 1.4).

Hence the net acceleration $\ddot{r} = \partial^2 r/\partial t^2$ of a (cylindrical) gas element with mass

$$dm = \rho \, dr \, dS \tag{2.7}$$

(where dr is its radial extent and dS is its horizontal surface area, see Fig. 2.1) is given by

$$\ddot{r} \, dm = -g \, dm + P(r) \, dS - P(r + dr) \, dS. \tag{2.8}$$

We can write $P(r + dr) = P(r) + (dP/dr) \cdot dr$, hence after substituting eqs. (2.6) and (2.7) we obtain the *equation of motion* for a gas element inside the star:

$$\ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{dP}{dr}. \tag{2.9}$$

Writing the pressure gradient in terms of the mass coordinate m by substituting eq. (2.5), the equation of motion is

$$\ddot{r} = -\frac{Gm}{r^2} - 4\pi r^2 \frac{dP}{dm}. \tag{2.10}$$

Hydrostatic equilibrium The great majority of stars are obviously in such long-lived phases of evolution that no change can be observed over human lifetimes. This means there is no noticeable acceleration, and all forces acting on a gas element inside the star almost exactly balance each other. Thus most stars are in a state of mechanical equilibrium which is more commonly called *hydrostatic equilibrium* (HE).

The state of hydrostatic equilibrium, setting $\ddot{r} = 0$ in eq. (2.9), yields the second differential equation of stellar structure:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho, \tag{2.11}$$

or with eq. (2.5)

$$\boxed{\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}} \tag{2.12}$$

A direct consequence is that inside a star in hydrostatic equilibrium, the pressure always decreases outwards.

Eqs. (2.5) and (2.12) together determine the *mechanical structure* of a star in HE. These are two equations for three unknown functions of m (r, P and ρ), so they cannot be solved without a third condition. This condition is usually a relation between P and ρ called the *equation of state* (see Chapter 3). In general the equation of state depends on the temperature T as well, so that the mechanical structure depends also on the temperature distribution inside the star, i.e. on its thermal structure. In special cases the equation of state is independent of T, and can be written as $P = P(\rho)$. In such cases (known as barotropes or polytropes) the mechanical structure of a star becomes independent of its thermal structure. This is the case for white dwarfs, as we shall see later.

Estimates of the central pressure A rough order-of-magnitude estimate of the central pressure can be obtained from eq. (2.12) by setting

$$\frac{dP}{dm} \sim \frac{P_{\rm surf} - P_c}{M} \approx -\frac{P_c}{M}, \quad m \sim \tfrac{1}{2}M, \quad r \sim \tfrac{1}{2}R$$

which yields

$$P_c \sim \frac{2}{\pi}\frac{GM^2}{R^4} \tag{2.13}$$

For the Sun we obtain from this estimate $P_c \sim 7 \times 10^{15}\,{\rm dyn/cm^2} = 7 \times 10^9$ atm.

A lower limit on the central pressure may be derived by writing eq. (2.12) as

$$\frac{dP}{dr} = -\frac{Gm}{4\pi r^4}\frac{dm}{dr} = -\frac{d}{dr}\left(\frac{Gm^2}{8\pi r^4}\right) - \frac{Gm^2}{2\pi r^5},$$

and thus

$$\frac{d}{dr}\left(P + \frac{Gm^2}{8\pi r^4}\right) = -\frac{Gm^2}{2\pi r^5} < 0. \tag{2.14}$$

The quantity $\Psi(r) = P + Gm^2/(8\pi r^4)$ is therefore a decreasing function of r. At the centre, the second term vanishes because $m \propto r^3$ for small r, and hence $\Psi(0) = P_c$. At the surface, the pressure is essentially zero. From the fact that Ψ must decrease with r it thus follows that

$$P_c > \frac{1}{8\pi}\frac{GM^2}{R^4}. \tag{2.15}$$

In contrast to eq. (2.13), this is a strict mathematical result, valid for any star in hydrostatic equilibrium regardless of its other properties (in particular, regardless of its density distribution). For the Sun we obtain $P_c > 4.4 \times 10^{14}\,{\rm dyn/cm^2}$. Both estimates indicate that an extremely high central pressure is required to keep the Sun in hydrostatic equilibrium. Realistic solar models show the central density to be $2.4 \times 10^{17}\,{\rm dyn/cm^2}$.

2.2.1 The dynamical timescale

We can ask what happens if the state of hydrostatic equilibrium is violated: how fast do changes to the structure of a star occur? The answer is provided by the equation of motion, eq. (2.9). For example, suppose that the pressure gradient that supports the star against gravity suddenly drops. All mass shells are then accelerated inwards by gravity: the star starts to collapse in “free fall”. We can approximate the resulting (inward) acceleration by

$$|\ddot{r}| \approx \frac{R}{{\tau_{\rm ff}}^2} \quad \Rightarrow \quad \tau_{\rm ff} \approx \sqrt{\frac{R}{|\ddot{r}|}}$$

where $\tau_{\rm ff}$ is the free-fall timescale that we want to determine. Since $-\ddot{r} = g \approx GM/R^2$ for the entire star, we obtain

$$\tau_{\rm ff} \approx \sqrt{\frac{R}{g}} \approx \sqrt{\frac{R^3}{GM}}. \tag{2.16}$$

Of course each mass shell is accelerated at a different rate, so this estimate should be seen as an average value for the star to collapse over a distance R. This provides one possible estimate for the *dynamical timescale* of the star. Another estimate can be obtained in a similar way by assuming that gravity suddenly disappears: this gives the timescale for the outward pressure gradient to explode the star, which is similar to the time it takes for a sound wave to travel from the centre to the surface of the star. If the star is close to HE, all these timescales have about the same value given by eq. (2.16). Since the average density $\bar{\rho} = 3M/(4\pi R^3)$, we can also write this (hydro)dynamical timescale as

$$\tau_{\rm dyn} \approx \sqrt{\frac{R^3}{GM}} \approx \tfrac{1}{2}\,(G\bar{\rho})^{-1/2}. \tag{2.17}$$

For the Sun we obtain a very small value of $\tau_{\rm dyn} \approx 1600$ sec or about half an hour (0.02 days). This is very much smaller than the age of the Sun, which is 4.6 Gyr or $\sim 1.5 \times 10^{17}$ sec, by 14 orders of magnitude. This result has several important consequences for the Sun and other stars:

- Any significant departure from hydrostatic equilibrium should very quickly lead to observable phenomena: either contraction or expansion on the dynamical timescale. If the star cannot recover from this disequilibrium by restoring HE, it should lead to a collapse or an explosion.

- Normally hydrostatic equilibrium can be restored after a disturbance (we will consider this *dynamical stability* of stars later). However a perturbation of HE may lead to small-scale oscillations on the dynamical timescale. These are indeed observed in the Sun and many other stars, with a period of minutes in the case of the Sun. Eq. (2.17) tells us that the pulsation period is a (rough) measure of the average density of the star.

- Apart from possible oscillations, stars are extremely close to hydrostatic equilibrium, since any disturbance is immediately quenched. We can therefore be confident that eq. (2.12) holds throughout most of their lifetimes. Stars do evolve and are therefore not completely static, but changes occur very slowly compared to their dynamical timescale. Stars can be said to evolve *quasi-statically*, i.e. through a series of near-perfect HE states.

2.3 The virial theorem

An important consequence of hydrostatic equilibrium is the *virial theorem*, which is of vital importance for the understanding of stars. It connects two important energy reservoirs of a star and allows predictions and interpretations of important phases in the evolution of stars.

To derive the virial theorem we start with the equation for hydrostatic equilibrium eq. (2.12). We multiply both sides by the enclosed volume $V = \frac{4}{3}\pi r^3$ and integrate over m:

$$\int_0^M \tfrac{4}{3}\pi r^3 \frac{dP}{dm} dm = -\tfrac{1}{3}\int_0^M \frac{Gm}{r} dm \tag{2.18}$$

The integral on the right-hand side has a straightforward physical interpretation: it is the *gravitational potential energy* of the star, i.e. the energy required to assemble the star by bringing matter from infinity,

$$E_{\rm gr} = -\int_0^M \frac{Gm}{r} dm \tag{2.19}$$

The left-hand side of eq. (2.18) can be integrated by parts:

$$\int_{P_c}^{P_s} V dP = [V \cdot P]_{\rm c}^{\rm s} - \int_0^{V_s} P dV \tag{2.20}$$

where c and s denote central and surface values. The first term vanishes because $V = 0$ at the centre and the pressure practically vanishes at the surface, $P(M) \approx 0$. Combining the above equations we obtain

$$-3 \int_0^{V_s} P dV = E_{\rm gr}, \tag{2.21}$$

or, since $dV = dm/\rho$,

$$-3 \int_0^{M} \frac{P}{\rho} dm = E_{\rm gr}. \tag{2.22}$$

This is the general form of the virial theorem, which will prove valuable later. It tells us that that the average pressure needed to support a star in HE is equal to $-\frac{1}{3}E_{\rm gr}/V$. In particular it tells us that a star that contracts quasi-statically (that is, slowly enough to remain in HE) must increase its internal pressure, since $|E_{\rm gr}|$ increases while its volume decreases.

The virial theorem for an ideal gas The pressure of a gas is related to its internal energy. We will show this in Ch. 3, but for the particular case of an ideal monatomic gas it is easy to see. The pressure of an ideal gas is given by

$$P = nkT = \frac{\rho}{\mu m_{\rm u}} kT, \tag{2.23}$$

where $n = N/V$ is the number of particles per unit volume, and μ is mass of a gas particle in atomic mass units. The kinetic energy per particle is $\epsilon_k = \frac{3}{2}kT$, and the internal energy of an ideal monatomic gas is equal to the kinetic energy of its particles. The internal energy per unit mass is then

$$u = \frac{3}{2}\frac{kT}{\mu m_{\rm u}} = \frac{3}{2}\frac{P}{\rho}. \tag{2.24}$$

We can now interpret the left-hand side of the virial theorem (eq. 2.22) as $\int (P/\rho) dm = \frac{2}{3}\int u dm = \frac{2}{3}E_{\rm int}$, where $E_{\rm int}$ is the total internal energy of the star. The virial theorem for an ideal gas is therefore

$$\boxed{E_{\rm int} = -\tfrac{1}{2}E_{\rm gr}} \tag{2.25}$$

This important relation establishes a link between the gravitational potential energy and the internal energy of a star in hydrostatic equilibrium that consists of an ideal gas. (We shall see later that the ideal gas law indeed holds for most stars, at least on the main sequence.) The virial theorem tells us that a more tightly bound star must have a higher internal energy, i.e. it must be *hotter*. In other words, a star that contracts quasi-statically must get hotter in the process. The full implications of this result will become clear when we consider the total energy of a star in a short while.

Estimate of the central temperature Using the virial theorem we can obtain an estimate of the average temparature inside a star composed of ideal gas. The gravitational energy of the star is found from eq. (2.19) and can be written as

$$E_{\rm gr} = -\alpha \frac{GM^2}{R}, \tag{2.26}$$

where α is a constant of order unity (determined by the distribution of matter in the star, i.e. by the density profile). Using eq. (2.24), the internal energy of the star is $E_{\rm int} = \frac{3}{2}k/(\mu m_{\rm u})\int T dm =$

$\frac{3}{2}k/(\mu m_\mathrm{u})\bar{T}M$, where $\bar{T}$ is the temperature averaged over all mass shells. By the virial theorem we then obtain

$$\bar{T} = \frac{\alpha}{3}\frac{\mu m_\mathrm{u}}{k}\frac{GM}{R}. \tag{2.27}$$

Taking $\alpha \approx 1$ and $\mu = 0.5$ for ionized hydrogen, we obtain for the Sun $\bar{T} \sim 4 \times 10^6$ K. This is the average temperature required to provide the pressure that is needed to keep the Sun in hydrostatic equilibrium. Since the temperature in a star normally decreses outwards, it is also an approximate lower limit on the central temperature of the Sun. At these temperatures, hydrogen and helium are indeed completely ionized. We shall see that $T_c \approx 10^7$ K is high enough for hydrogen fusion to take place in the central regions of the Sun.

The virial theorem for a general equation of state Also for equations of state other than an ideal gas a relation between pressure and internal energy exists, which we can write generally as

$$u = \phi\frac{P}{\rho}. \tag{2.28}$$

We have seen above that $\phi = \frac{3}{2}$ for an ideal gas, but it will turn out (see Ch. 3) that this is valid not only for an ideal gas, but for all non-relativistic particles. On the other hand, if we consider a gas of relativistic particles, in particular photons (i.e. radiation pressure), $\phi = 3$. If ϕ is constant throughout the star we can integrate the left-hand side of eq. (2.21) to obtain a more general form of the virial theorem:

$$E_\mathrm{int} = -\tfrac{1}{3}\phi E_\mathrm{gr} \tag{2.29}$$

2.3.1 The total energy of a star

The total energy of a star is the sum of its gravitational potential energy, its internal energy and its kinetic energy E_kin (due to bulk motions of gas inside the star, not the thermal motions of the gas particles):

$$E_\mathrm{tot} = E_\mathrm{gr} + E_\mathrm{int} + E_\mathrm{kin}. \tag{2.30}$$

The star is bound as long as its total energy is negative.

For a star in hydrostatic equilibrium we can set $E_\mathrm{kin} = 0$. Furthermore for a star in HE the virial theorem holds, so that E_gr and E_int are tightly related by eq. (2.29). Combining eqs. (2.29) and (2.30) we obtain the following relations:

$$E_\mathrm{tot} = E_\mathrm{int} + E_\mathrm{gr} = \frac{\phi - 3}{\phi}E_\mathrm{int} = (1 - \tfrac{1}{3}\phi)E_\mathrm{gr} \tag{2.31}$$

As long as $\phi < 3$ the star is bound. This is true in particular for the important case of a star consisting of an ideal gas (eq. 2.25, for which we obtain

$$\boxed{E_\mathrm{tot} = E_\mathrm{int} + E_\mathrm{gr} = -E_\mathrm{int} = \tfrac{1}{2}E_\mathrm{gr} < 0} \tag{2.32}$$

In other words, its total energy of such a star equals half of its gravitational potential energy.

From eq. (2.32) we can see that the virial theorem has the following important consequences:

- Gravitationally bound gas spheres must be *hot* to maintain hydrostatic equilibrium: heat provides the pressure required to balance gravity. The more compact such a sphere, the more strongly bound, and therefore the hotter it must be.

- A hot sphere of gas radiates into surrounding space, therefore a star must lose energy from its surface. The rate at which energy is radiated from the surface is the *luminosity* of the star. In the absence of an internal energy source, this energy loss must equal the decrease of the total energy of the star: $L = -dE_{\rm tot}/dt > 0$, since L is positive by convention.

- Taking the time derivative of eq. (2.32), we find that as a consequence of losing energy:

  $$\dot{E}_{\rm gr} = -2L < 0,$$

  meaning that the star *contracts* (becomes more strongly bound), and

  $$\dot{E}_{\rm int} = L > 0,$$

  meaning that the star *gets hotter* – unlike familiar objects which cool when they lose energy. Therefore a star can be said to have a *negative heat capacity*. Half the energy liberated by contraction is used for heating the star, the other half is radiated away.

For the case of a star that is dominated by radiation pressure, we find that $E_{\rm int} = -E_{\rm gr}$, and therefore the total energy $E_{\rm tot} = 0$. Therefore a star dominated by radiation pressure (or more generally, by the pressure of relativistic particles) is only marginally bound. No energy is required to expand or contract such a star, and a small perturbation would be enough to render it unstable and to trigger its collapse or complete dispersion.

2.3.2 Thermal equilibrium

If internal energy sources are present in a star due to nuclear reactions taking place in the interior, then the energy loss from the surface can be compensated: $L = L_{\rm nuc} \equiv -dE_{\rm nuc}/dt$. In that case the total energy is conserved and eq. (2.32) tells us that $\dot{E}_{\rm tot} = \dot{E}_{\rm int} = \dot{E}_{\rm gr} = 0$. The virial theorem therefore tells us that both $E_{\rm int}$ and $E_{\rm gr}$ are conserved as well: the star cannot, for example, contract and cool while keeping its total energy constant.

In this state, known as *thermal equilibrium* (TE), the star is in a stationary state: it neither expands nor contracts, and it maintains a constant interior temperature. We shall see later that this temperature is regulated by the nuclear reactions themselves, which in combination with the virial theorem act like a stellar thermostat. Main-sequence stars like the Sun are in thermal equilibrium, and a star can remain in this state as long as nuclear reactions can supply the necessary energy.

2.4 Local energy conservation

The previous section dealt with the global energy budget of a star. We have still to consider the conservation of energy on a local scale in the stellar interior, which leads us the third stellar structure equation.

The first law of thermodynamics states that the internal energy of a system can be changed by two forms of energy transfer: heat and work. Heat may be added or extracted, and work may be done on the system or performed by the system, and involves a change of its volume: expansion or contraction. Consider a spherical, Lagrangian shell inside the star of constant mass dm. By δf we denote a change in a quantity f occurring in a small time interval δt. If u is the internal energy per unit mass, then the first law can be written as

$$\delta u \, dm = \delta Q + \delta W \qquad (2.33)$$

where δQ is the amount of heat absorbed or emitted by the shell, and δW is the work done on the shell,

$$\delta W = -P\delta(dV) = -P\,\delta\left(\frac{1}{\rho}\right)dm = \frac{P}{\rho^2}\,\delta\rho\,dm. \tag{2.34}$$

In the last equation $dV = dm/\rho$ is the volume of the mass element, and dm is constant. We note that compression ($\delta\rho > 0$) involves an addition of energy, and expansion is achieved at the expense of the element's own energy.

Changes in the heat content of the shell δQ can occur due to a number of sources and sinks:

- Heat is added by the release of nuclear energy, if available. The rate at which nuclear energy is produced per unit mass and per second is written as $\epsilon_{\rm nuc}$. The details of nuclear energy generation will be treated in Ch. 5.
- Heat can be removed by the release of energetic neutrinos, which escape from the stellar interior without interaction. Neutrinos are released as a by-product of some nuclear reactions, in which case they are accounted for in $\epsilon_{\rm nuc}$. But neutrinos can also be released by weak interaction processes in very hot and dense plasmas. This type of neutrino production plays a role in late phases of stellar evolution, and the rate at which these neutrinos take away energy per unit mass is written as ϵ_ν.
- Finally, heat is absorbed or emitted according to the balance of heat fluxes flowing into and out of the shell (see Fig. 2.2). We define a new variable, the *local luminosity* l, as the rate at which energy in the form of heat flows outward through a sphere of radius r. Therefore at the surface $l = L$ while at the centre $l = 0$. Normally heat flows outwards, in the direction of decreasing temperature. Therefore l is usually positive, but under some circumstances (e.g. cooling of central regions by neutrino emission) heat can flow inwards, meaning that l is negative.

We can therefore write:

$$\delta Q = \epsilon_{\rm nuc}\,dm\,\delta t - \epsilon_\nu\,dm\,\delta t + l(m)\,\delta t - l(m+dm)\,\delta t,$$

with $l(m+dm) = l(m) + (dl/dm)\cdot dm$, so that

$$\delta Q = \left(\epsilon_{\rm nuc} - \epsilon_\nu - \frac{dl}{dm}\right)dm\,\delta t. \tag{2.35}$$

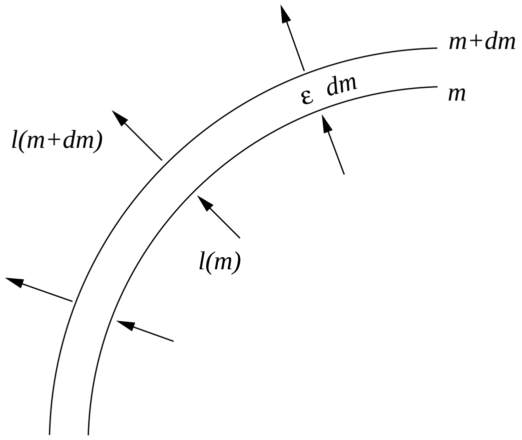


Figure 2.2. Energy generation and heat flow into and out of a spherical mass shell.

Substituting eqs. (2.34) and (2.35) into eq. (2.33) and dividing by *dm* we obtain

$$\delta u - \frac{P}{\rho^2}\,\delta\rho = \left(\epsilon_{\rm nuc} - \epsilon_\nu - \frac{dl}{dm}\right)\delta t,$$

which in the limit $\delta t \to 0$ yields us the third equation of stellar evolution:

$$\boxed{\frac{dl}{dm} = \epsilon_{\rm nuc} - \epsilon_\nu - \frac{\partial u}{\partial t} + \frac{P}{\rho^2}\frac{\partial\rho}{\partial t}} \tag{2.36}$$

In *thermal equilibrium* (see Sec. 2.3.2), the star is in a stationary state and the time derivatives vanish. We then obtain a much simpler stellar structure equation,

$$\frac{dl}{dm} = \epsilon_{\rm nuc} - \epsilon_\nu. \tag{2.37}$$

If we integrate this equation over the mass we obtain

$$L = \int_0^M \epsilon_{\rm nuc}\, dm - \int_0^M \epsilon_\nu\, dm \equiv L_{\rm nuc} - L_\nu \tag{2.38}$$

which defines the nuclear luminosity $L_{\rm nuc}$ and the neutrino luminosity L_ν. Neglecting the neutrino losses for the moment, we see that thermal equilibrium implies that $L = L_{\rm nuc}$, that is, energy is radiated away at the surface at the same rate at which it is produced by nuclear reactions in the interior.

2.5 The timescales of stellar evolution

Three important timescales are relevant for stellar evolution, associated with changes to the mechanical structure of a star (described by eq. 2.10), changes to its thermal structure (described by eq. 2.36), and changes in its composition, which are still to be discussed in Ch. 5.

The first timescale was already treated in Sec. 2.2.1: it is the *dynamical timescale* given by eq. (2.17),

$$\tau_{\rm dyn} \approx \sqrt{\frac{R^3}{GM}} \approx 0.02\left(\frac{R}{R_\odot}\right)^{3/2}\left(\frac{M_\odot}{M}\right)^{1/2} \text{days} \tag{2.39}$$

The dynamical timescale is the timescale on which a star reacts to a perturbation of hydrostatic equilibrium. We saw that this timescale is typically of the order of hours or less, which means that stars are extremely close to hydrostatic equilibrium.

2.5.1 The thermal timescale

The second timescale describes how fast changes in the thermal structure of a star can occur. It is therefore also the timescale on which a star in thermal equilibrium reacts when its TE is perturbed. To obtain an estimate, we turn to the virial theorem: we saw in Sec. 2.3.1 that a star without a nuclear energy source contracts by radiating away its internal energy content: $L = \dot{E}_{\rm int} \approx -2\dot{E}_{\rm gr}$, where the last equality applies strictly only for an ideal gas. We can thus define the *thermal* or *Kelvin-Helmholtz timescale* as the timescale on which this gravitational contraction would occur:

$$\tau_{\rm KH} = \frac{E_{\rm int}}{L} \approx \frac{|E_{\rm gr}|}{2L} \approx \frac{GM^2}{2RL} \approx 1.5\times10^7\left(\frac{M}{M_\odot}\right)^2\frac{R_\odot}{R}\frac{L_\odot}{L}\ \text{yr} \tag{2.40}$$

Here we have used eq. 2.26 for $E_{\rm gr}$ with $\alpha \approx 1$.

The thermal timescale for the Sun is about 1.5×10^7 years, which is many orders of magnitude larger than the dynamical timescale. There is therefore no direct observational evidence that any star is in thermal equilibrium, and in the late 19th century (before the importance of nuclear reactions was recognized) gravitational contraction was proposed as the energy source of the Sun by Lord Kelvin and, independently, by Hermann von Helmholtz. Nuclear reactions have since turned out to be a much more powerful energy source, allowing stars to be in thermal equilibrium for most ($> 99\,\%$) of their lifetimes. However, several phases of stellar evolution, during which the nuclear power source is absent or inefficient, do occur on the thermal timescale.

2.5.2 The nuclear timescale

A star can remain in thermal equilibrium for as longs as its nuclear fuel supply lasts. The associated timescale is called the *nuclear timescale*, and since nuclear fuel (say hydrogen) is burned into 'ash' (say helium), it is also the timescale on which composition changes in the stellar interior occur.

The energy source of nuclear fusion is the direct conversion of a small fraction ϕ of the rest mass of the reacting nuclei into energy. For hydrogen fusion, $\phi \approx 0.007$; for fusion of helium and heavier elements ϕ is smaller by a factor 10 or more. The total nuclear energy supply can therefore be written as $E_{\rm nuc} = \phi M_{\rm nuc} c^2 = \phi f_{\rm nuc} M c^2$, where $f_{\rm nuc}$ is that fraction of the mass of the star which may serve as nuclear fuel. In thermal equilibrium $L = L_{\rm nuc} = \dot{E}_{\rm nuc}$, so we can estimate the nuclear timescale as

$$\tau_{\rm nuc} = \frac{E_{\rm nuc}}{L} = \phi f_{\rm nuc} \frac{Mc^2}{L} \approx 10^{10} \frac{M}{M_\odot} \frac{L_\odot}{L}\ \text{yr}. \tag{2.41}$$

The last approximate equality holds for hydrogen fusion in a star like the Sun, with has 70 % of its initial mass in hydrogen and fusion occurring only in the inner $\approx$ 10 % of its mass (the latter result comes from detailed stellar models). This long timescale is consistent with the geological evidence for the age of the Earth.

We see that, despite only a small fraction of the mass being available for fusion, the nuclear timescale is indeed two to three orders of magnitude larger than the thermal timescale. Therefore the assumption that stars can reach a state of thermal equilibrium is justified. To summarize, we have found:

$$\tau_{\rm nuc} \gg \tau_{\rm KH} \gg \tau_{\rm dyn}.$$

As a consequence, the rates of nuclear reactions determine the pace of stellar evolution, and stars may be assumed to be in hydrostatic and thermal equilibrium throughout most of their lives.

Suggestion for further reading

The contents of this chapter are covered more extensively by Chapters 1 to 4 of KIPPENHAHN & WEIGERT.

Chapter 3

Equation of state of stellar interiors

3.1 Local thermodynamic equilibrium

Empirical evidence shows that in a part of space isolated from the rest of the Universe, matter and radiation tend towards a state of *thermodynamic equilibrium*. This equilibrium state is achieved when sufficient interactions take place between the material particles (‘collisions’) and between the photons and mass particles (scatterings and absorptions). In such a state of thermodynamic equilibrium the radiation field becomes isotropic and the photon energy distribution is described by the Planck function (blackbody radiation). The statistical distribution functions of both the mass particles and the photons are then characterized by a single temperature T.

We know that stars are not isolated systems, because they emit radiation and generate (nuclear) energy in their interiors. Indeed, the surface temperature of the Sun is about 6000 K, while we have estimated from the virial theorem (Sec. 2.3) that the interior temperature must of the order of 10^7 K. Therefore stars are *not* in global thermodynamic equilibrium. However, it turns out that locally within a star, a state of thermodynamic equilibrium *is* achieved. This means that within a region much smaller than the dimensions of a star ($\ll R_*$), but larger than the average distance between interactions of the particles (both gas particles and photons), i.e. larger than the mean free path, there is a well-defined *local temperature* that describes the particle statistical distributions.

We can make this plausible by considering the mean free path for photons:

$$\ell_{\rm ph} = 1/\kappa\rho$$

where κ is the opacity coefficient, i.e. the effective cross section per unit mass. For fully ionized matter, a minimum is given by the electron scattering cross section, which is $\kappa_{\rm es} = 0.4\,{\rm cm}^2/{\rm g}$ (see Ch. 4). The average density in the Sun is $\bar{\rho} = 1.4\,{\rm g/cm}^3$, which gives a mean free path of the order of $\ell_{\rm ph} \sim 1$ cm. In other words, stellar matter is very opaque to radiation. The temperature difference over a distance $\ell_{\rm ph}$, i.e. between emission and absorption, can be estimated as

$$\Delta T \approx \frac{dT}{dr}\ell_{\rm ph} \approx \frac{T_c}{R}\ell_{\rm ph} \approx \frac{10^7}{10^{11}} \approx 10^{-4}\,{\rm K}$$

which is a tiny fraction (10^{-11}) of the typical interior temperature of 10^7 K. Using a similar estimate, it can be shown that the mean free path for interactions between ionized gas particles (ions and electrons) is several orders of magnitude smaller than $\ell_{\rm ph}$. Hence a small region can be defined (a ‘point’ for all practical purposes) which is $> \ell_{\rm ph}$ but much smaller than the length scale over which significant changes of thermodynamic quantities occur. This is called *local thermodynamic equilibrium* (LTE). We can therefore assume a well-defined temperature distribution inside the star.

Furthermore, the average time between particle interactions (the mean free time) is much shorter than the timescale for changes of the macroscopic properties. Therefore a state of LTE is secured at all times in the stellar interior. The assumption of LTE[1] constitutes a great simplification. It enables the calculation of all thermodynamic properties of the stellar gas in terms of the local values of temperature, density and composition, as they change from the centre to the surface.

3.2 The equation of state

The equation of state (EOS) describes the microscopic properties of stellar matter, for given density ρ, temperature T and composition X_i. It is usually expressed as the relation between the pressure and these quantities:

$$\boxed{P = P(\rho, T, X_i)} \tag{3.1}$$

Using the laws of thermodynamics, and a similar equation for the internal energy $U(\rho, T, X_i)$, we can derive from the EOS the thermodynamic properties that are needed to describe the structure of a star, such as the specific heats c_V and c_P, the adiabatic exponent γ_{ad} and the adiabatic temperature gradient ∇_{ad}.

An example is the ideal-gas equation of state, which in the previous chapters we have tacitly assumed to hold for stars like the Sun:

$$P = nkT \quad \text{or} \quad P = \frac{k}{\mu m_{\mathrm{u}}}\rho T.$$

In this chapter we will see whether this assumption was justified, and how the EOS can be extended to cover all physical conditions that may prevail inside a star. The ideal-gas law pertains to particles that behave according to classical physics. However, both quantum-mechanical and special relativistic effects may be important under the extreme physical conditions in stellar interiors. In addition, photons (which can be described as extremely relativistic particles) can be an important source of pressure.

We can define an ideal or *perfect* gas as a mixture of free, non-interacting particles. Of course the particles in such a gas do interact, so more precisely we require that their interaction energies are small compared to their kinetic energies. In that case the internal energy of the gas is just the sum of all kinetic energies. From statistical mechanics we can derive the properties of such a perfect gas, both in the classical limit (recovering the ideal-gas law) and in the quantum-mechanical limit (leading to electron degeneracy), and both in the non-relativistic and in the relativistic limit (e.g. valid for radiation).

In addition, various *non-ideal* effects may become important. The high temperatures ($> 10^6$ K) in stellar interiors ensure that the gas will be fully ionized, but at lower temperatures (in the outer layers) partial ionization has to be considered, with important effects on the thermodynamic properties (see Sect. 3.5). Furthermore, in an ionized gas *electrostatic interactions* between the ions and electrons may be important under certain circumstances.

3.2.1 The perfect gas approximation

Is the perfect gas approximation reasonable in stellar interiors, i.e. are the interaction energies indeed small compared to the kinetic energies?

---

[1] N.B. note the difference between (local) *thermodynamic equilibrium* ($T_{\mathrm{gas}}(r) = T_{\mathrm{rad}}(r) = T(r)$) and the earlier defined, global property of *thermal equilibrium* ($E_{\mathrm{tot}} = $ const, or $L = L_{\mathrm{nuc}}$).

The average distance between gas particles (with mass Am_u) is $d \approx (\frac{4\pi}{3}n)^{-1/3}$ where n is the number density, $n = \rho/(Am_\mathrm{u})$. The typical Coulomb energy per particle (with charge Ze) is $\epsilon_\mathrm{C} \approx Z^2e^2/d$, so that the ratio of Coulomb energy to average kinetic energy, $\epsilon_\mathrm{kin} = \frac{3}{2}kT$, is

$$\frac{\epsilon_\mathrm{C}}{\epsilon_\mathrm{kin}} \approx \frac{Z^2e^2}{d\,kT} \approx \frac{Z^2e^2}{(Am_\mathrm{u})^{1/3}}\frac{\rho^{1/3}}{kT}. \tag{3.2}$$

We see that Coulomb interactions increase in importance at high densities or low temperatures. To estimate the ratio $\epsilon_\mathrm{C}/\epsilon_\mathrm{kin}$ in stellar interiors we approximate $\rho \approx \bar{\rho} = M/(\frac{4\pi}{3}R^3)$, and we approximate T by the average temperature estimated from the virial theorem, $T \approx \bar{T} \approx \frac{1}{3}\frac{Am_\mathrm{u}}{k}\frac{GM}{R}$. Ignoring factors of order unity, we get

$$\frac{\epsilon_\mathrm{C}}{\epsilon_\mathrm{kin}} \approx \frac{Z^2e^2}{(Am_\mathrm{u})^{4/3}G}M^{-2/3} \approx 0.01\frac{Z^2}{A^{4/3}}\left(\frac{M}{M_\odot}\right)^{-2/3}. \tag{3.3}$$

The ratio $Z^2/A^{4/3}$ depends on the composition, and represents an average over the constituents of the gas. In stars mostly composed of hydrogen, $A \approx 1$ and $Z \approx 1$, and we find that in the Sun the Coulomb energy contributes of the order of 1 % to the particle energies (and hence has a similar effect on the pressure). We are therefore justified in ignoring Coulomb interactions in stars, certainly in those similar or more massive than the Sun. However, eq. (3.3) shows that in low-mass stars Coulomb interactions can start to contribute significantly, and detailed models of such stars need to take this into account. For $M \lesssim 10^{-3}\,M_\odot$ the Coulomb energies dominate, however, such objects are not stars but planets (Jupiter's mass is about $10^{-3}\,M_\odot$). Calculations of the structure of planets requires a much more complicated equation of state than for stars.

3.3 The equation of state for a gas of free particles

We shall derive the equation of state for a perfect gas from the principles of statistical mechanics. This provides a description of the ions, the electrons, as well as the photons in the deep stellar interior.

Let $n(p)$ be the distribution of momenta of the gas particles, i.e. $n(p)dp$ represents the number of particles per unit volume with momenta $p \in [p \ldots p + dp]$. If $n(p)$ is known then the total number density (number of particles per unit volume), the internal energy density (internal energy per unit volume) and the pressure can be obtained from the following integrals:

$$\text{number density} \qquad n = \int_0^\infty n(p)dp \tag{3.4}$$

$$\text{internal energy density} \qquad U = \int_0^\infty \epsilon_p n(p)dp = n\langle\epsilon_p\rangle \tag{3.5}$$

$$\text{pressure} \qquad P = \tfrac{1}{3}\int_0^\infty p v_p n(p)dp = \tfrac{1}{3}n\langle p v_p\rangle \tag{3.6}$$

Here ϵ_p is the kinetic energy of a particle with momentum p, and v_p is its velocity. Eq. (3.4) is trivial, and eq. (3.5) follows from the perfect-gas assumption. The pressure integral eq. (3.6) requires some explanation.

Consider a gas of n particles in a cubical box with sides of length $L = 1$ cm. Each particle bounces around in the box, and the pressure on one side of the box results from the momentum imparted by all the particles colliding with it. Consider a particle with momentum p and corresponding velocity v coming in at an angle θ with the normal to the surface, as depicted in Fig. 3.1. The time between two collisions with the same side is

$$\Delta t = \frac{2L}{v\cos\theta} = \frac{2}{v\cos\theta}.$$

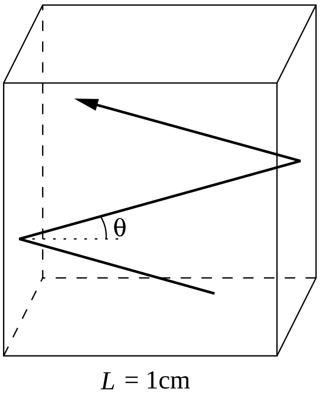


Figure 3.1. Gas particle in a cubical box with a volume of $1\,\mathrm{cm}^3$. Each collision with the side of the box results in a transfer of momentum; the pressure inside the box is the result of the collective momentum transfers of all n particles in the box.

The collisions are elastic, so the momentum transfer is twice the momentum component perpendicular to the surface,

$$\Delta p = 2p\cos\theta. \tag{3.7}$$

The momentum transferred per particle per second and per cm^2 is therefore

$$\frac{\Delta p}{\Delta t} = vp\ \cos^2\theta. \tag{3.8}$$

The number of particles in the box with $p \in [p \ldots p + dp]$ and $\theta \in [\theta \ldots \theta + d\theta]$ is denoted as $n(\theta, p)\, d\theta\, dp$. The contribution to the pressure from these particles is then

$$dP = vp\ \cos^2\theta\, n(\theta, p)\, d\theta\, dp. \tag{3.9}$$

Since the momenta are distributed isotropically over all directions within a solid angle 2π, and the solid angle $d\omega$ subtended by those particles with $\theta \in [\theta \ldots \theta + d\theta]$ equals $2\pi\sin\theta\, d\theta$, we have $n(\theta, p)\, d\theta = n(p)\, sin\theta\, d\theta$ and

$$dP = vp\, n(p)\ \cos^2\theta \sin\theta\, d\theta\, dp. \tag{3.10}$$

The total pressure is obtained by integrating over all angles ($0 \leq \theta \leq \pi/2$) and momenta. This results in eq. (3.6) since $\int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta = \int_0^1 \cos^2\theta\, d\cos\theta = \frac{1}{3}$.

3.3.1 Relation between pressure and internal energy

In general, the particle energies and velocities are related to their momenta according to special relativity:

$$\epsilon^2 = p^2c^2 + m^2c^4, \qquad \epsilon_p = \epsilon - mc^2 \tag{3.11}$$

and

$$v_p = \frac{\partial\epsilon}{\partial p} = \frac{pc^2}{\epsilon}. \tag{3.12}$$

We can obtain generally valid relations between the pressure and the internal energy of a perfect gas in the non-relativistic (NR) limit and the extremely relativistic (ER) limit:

NR limit: in this case the momenta $p \ll mc$, so that $\epsilon_p = \epsilon - mc^2 = \frac{1}{2}p^2/m$ and $v = p/m$. Therefore $\langle pv\rangle = \langle p^2/m\rangle = 2\langle\epsilon_p\rangle$ so that eq. (3.6) yields

$$\boxed{P = \tfrac{2}{3}U} \tag{3.13}$$

ER limit: in this case $p \gg mc$, so that $\epsilon_p = pc$ and $v = c$. Therefore $\langle pv \rangle = \langle pc \rangle = \langle \epsilon_p \rangle$, and eq. (3.6) yields

$$\boxed{P = \tfrac{1}{3}U} \tag{3.14}$$

These relations are generally true, for *any particle* (electrons, ions and photons). We will apply this in the coming sections. As we saw in the previous Chapter, the change from $\frac{2}{3}$ to $\frac{1}{3}$ in the relation has important consequences for the virial theorem, and for the stability of stars.

3.3.2 The classical ideal gas

Using the tools of statistical mechanics, we can address the origin of the ideal-gas law. The momentum distribution $n(p)$ for classical, non-relativistic particles of mass m in LTE is given by the *Maxwell-Boltzmann* distribution:

$$n(p)dp = \frac{n}{(2\pi mkT)^{3/2}}\, e^{-p^2/2mkT}\, 4\pi p^2\, dp. \tag{3.15}$$

Here the exponential factor ($e^{-\epsilon_p/kT}$) represents the equilibrium distribution of kinetic energies, the factor $4\pi p^2\, dp$ is the volume in momentum space (p_x, p_y, p_z) for $p \in [p \ldots p + dp]$, and the factor $n/(2\pi mkT)^{3/2}$ comes from the normalization of the total number density n imposed by eq. (3.4). (You can verify this by starting from the standard integral $\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\pi/a}$, and differentiating once with respect to a to obtain the integral $\int_0^\infty e^{-ax^2} x^2 dx$.)

The pressure is calculated by using $v = p/m$ for the velocity in eq. (3.6):

$$P = \frac{1}{3}\frac{n}{(2\pi mkT)^{3/2}} \int_0^\infty \frac{p^2}{m}\, e^{-p^2/2mkT}\, 4\pi p^2 dp. \tag{3.16}$$

By performing the integration (for this you need to differentiate $\int_0^\infty e^{-ax^2} x^2 dx$ once more with respect to a) you can verify that this indeed yields the ideal gas law

$$\boxed{P = nkT}. \tag{3.17}$$

(N.B. This derivation is for a gas of *non-relativistic* classical particles, but it can be shown that the same relation $P = nkT$ is also valid for *relativistic* classical particles.)

3.3.3 Mixture of ideal gases, and the mean molecular weight

The ideal gas relation was derived for identical particles of mass m. It should be obvious that for a mixture of free particles of different species, it holds for the partial pressures of each of the constituents of the gas separately. In particular, it holds for both the ions and the electrons, as long as quantum-mechanical effects can be ignored. The total gas pressure is then just the sum of partial pressures

$$P_{\rm gas} = P_{\rm ion} + P_e = \textstyle\sum_i P_i + P_e = (\sum_i n_i + n_e)kT = nkT$$

where n_i is the number density of ions of element i, with mass $m_i = A_i m_{\rm u}$ and charge $Z_i e$. Then n_i is related to the density and the mass fraction X_i of this element as

$$n_i = \frac{X_i\, \rho}{A_i\, m_{\rm u}} \qquad \text{and} \qquad n_{\rm ion} = \sum_i \frac{X_i}{A_i}\frac{\rho}{m_{\rm u}} \equiv \frac{1}{\mu_{\rm ion}}\frac{\rho}{m_{\rm u}}, \tag{3.18}$$

which defines the mean atomic mass per ion $\mu_{\rm ion}$. The partial pressure due to all ions is then

$$P_{\rm ion} = \frac{1}{\mu_{\rm ion}} \frac{\rho}{m_{\rm u}} kT = \frac{\mathcal{R}}{\mu_{\rm ion}} \rho T. \tag{3.19}$$

We have used here the universal gas constant $\mathcal{R} = k/m_{\rm u} = 8.31447 \times 10^7\,{\rm erg\,g^{-1}\,K^{-1}}$. The number density of electrons is given by

$$n_e = \sum_i Z_i n_i = \sum_i \frac{Z_i X_i}{A_i} \frac{\rho}{m_{\rm u}} \equiv \frac{1}{\mu_e} \frac{\rho}{m_{\rm u}}, \tag{3.20}$$

which defines the *mean molecular weight per free electron* μ_e. As long as the electrons behave like classical particles, the electron pressure is thus given by

$$P_e = \frac{1}{\mu_e} \frac{\rho}{m_{\rm u}} kT = \frac{\mathcal{R}}{\mu_e} \rho T. \tag{3.21}$$

When the gas is fully ionized, we have for hydrogen $Z_i = A_i = 1$ while for helium and the most abundant heavier elements, $Z_i/A_i \approx \frac{1}{2}$. In terms of the hydrogen mass fraction X we then get

$$\mu_e \approx \frac{2}{1+X}, \tag{3.22}$$

which for the Sun ($X = 0.7$) amounts to $\mu_e \approx 1.18$, and for hydrogen-depleted gas gives $\mu_e \approx 2$.

The total gas pressure is then given by

$$P_{\rm gas} = P_{\rm ion} + P_e = \Big(\frac{1}{\mu_{\rm ion}} + \frac{1}{\mu_e}\Big) \mathcal{R} \rho T = \frac{\mathcal{R}}{\mu} \rho T \tag{3.23}$$

where the *mean molecular weight* μ is given by

$$\frac{1}{\mu} = \frac{1}{\mu_{\rm ion}} + \frac{1}{\mu_e} = \sum_i \frac{(Z_i+1) X_i}{A_i}. \tag{3.24}$$

It is left as an exercise to show that for a fully ionized gas, μ can be expressed in terms of the mass fractions X, Y and Z as

$$\mu \approx \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z} \tag{3.25}$$

if we assume that for elements heavier than helium, $A_i \approx 2Z_i \approx 2(Z_i + 1)$.

3.3.4 Quantum-mechanical description of the gas

According to quantum mechanics, the accuracy with which a particle's location and momentum can be known simultaneously is limited by Heisenberg's uncertainty principle, i.e. $\Delta x \Delta p \geq h$. In three dimensions, this means that if a particle is located within a volume element ΔV then its localization within three-dimensional momentum space $\Delta^3 p$ is constrained by

$$\Delta V\, \Delta^3 p \geq h^3. \tag{3.26}$$

The quantity h^3 defines the volume in six-dimensional phase space of one quantum cell. The *number of quantum states* in a spatial volume V and with momenta $p \in [p \ldots p + dp]$ is therefore given by

$$g(p)\, dp = g_s \frac{V}{h^3} 4\pi p^2 dp, \tag{3.27}$$

where g_s is the number of intrinsic quantum states of the particle, e.g. spin or polarization.

The relative occupation of the available quantum states for particles in thermodynamic equilibrium depends on the type of particle:

- *fermions* (e.g. electrons or nucleons) obey the Pauli exclusion principle, which postulates that no two such particles can occupy the same quantum state. The fraction of states with energy ϵ_p that will be occupied at temperature T is given by

  $$f_{\rm FD}(\epsilon_p) = \frac{1}{e^{(\epsilon_p-\mu)/kT} + 1}, \tag{3.28}$$

  which is always ≤ 1.

- *bosons* (e.g. photons) have no restriction on the number of particles per quantum state, and the fraction of states with energy ϵ_p that is occupied is

  $$f_{\rm BE}(\epsilon_p) = \frac{1}{e^{(\epsilon_p-\mu)/kT} - 1}, \tag{3.29}$$

  which can be > 1.

The actual distribution of momenta for particles in LTE is given by the product of the occupation fraction $f(\epsilon_p)$ and the number of quantum states, given by eq. (3.27). The quantity μ appearing in eqs. (3.28) and (3.29) is the so-called *chemical potential*. It can be seen as a normalization constant, determined by the total number of particles in the volume considered (i.e., by the constraint imposed by eq. 3.4).

3.3.5 Electron degeneracy

Electrons are fermions with two spin states, i.e. $g_e = 2$. According to eq. (3.27), the maximum number density of electrons with momentum p allowed by quantum mechanics is therefore

$$n_{\rm max}(p)\,dp = \frac{g_e}{h^3}\,4\pi p^2 dp = \frac{8\pi}{h^3}\,p^2 dp. \tag{3.30}$$

This is shown as the dotted line in Fig. 3.2. The actual momentum distribution of electrons $n_e(p)$ is given by the product of eq. (3.30) and eq. (3.28). In the non-relativistic limit we have $\epsilon_p = p^2/2m_e$, giving

$$n_e(p)\,dp = \frac{2}{h^3}\,\frac{1}{e^{(p^2/2m_ekT)-\psi} + 1}\,4\pi p^2 dp, \tag{3.31}$$

where we have replaced the chemical potential by the *degeneracy parameter* $\psi = \mu/kT$. The value of ψ is determined by the constraint that $\int_0^\infty n_e(p)\,dp = n_e$ (eq. 3.4).

The limitation imposed by the Pauli exclusion principle means that electrons can exert a higher pressure than predicted by classical physics (eq. 3.21). To illustrate this, in Fig. 3.2 the momentum distribution eq. (3.31) is compared to the Maxwell-Boltzmann distribution for electrons, eq. (3.15),

$$n_{\rm MB}(p)\,dp = \frac{n_e}{(2\pi m_e kT)^{3/2}}\,e^{-p^2/2m_ekT}\,4\pi p^2 dp. \tag{3.32}$$

The situation shown is for an electron density $n_e = 6 \times 10^{27}\,{\rm cm}^{-3}$, which corresponds to a mass density of $2 \times 10^4\,{\rm g/cm}^{-3}$ (assuming a hydrogen-depleted gas with $\mu_e = 2$). At high temperatures, $T = 2 \times 10^7$ K, the momentum distribution (solid line) nearly coincides with the M-B distribution (dashed line): none of the quantum states are fully occupied ($n_e(p) < n_{\rm max}(p)$ for all values of p) and the electrons behave like classical particles. As the temperature is decreased, e.g. at $T = 2 \times 10^6$ K (red lines), the peak in the M-B distribution shifts to smaller p and is higher (since the integral over

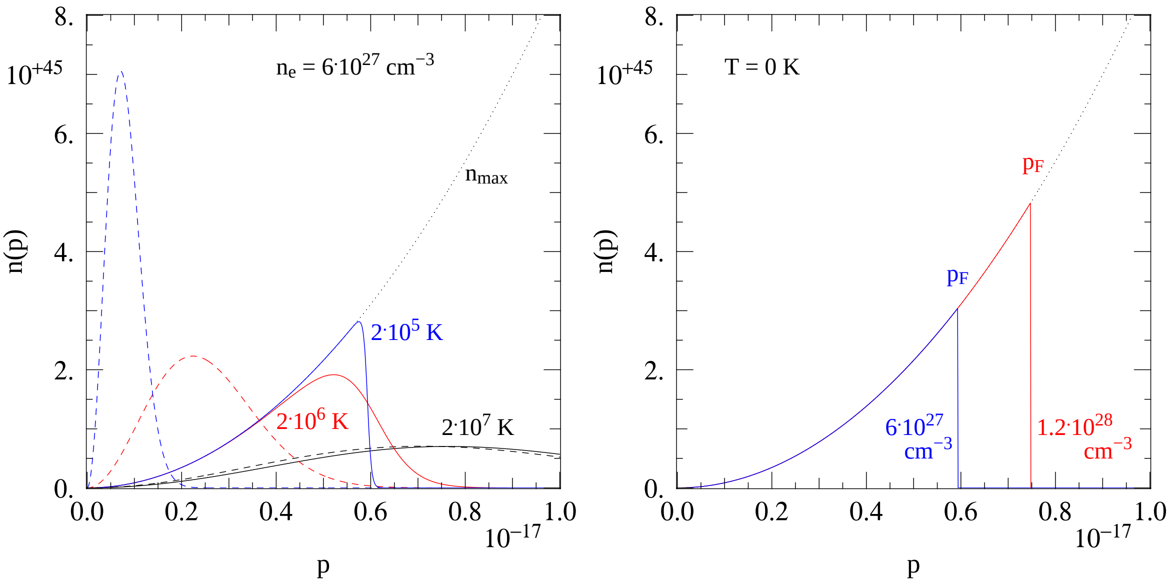


Figure 3.2. (left panel) Electron momentum distributions $n(p)$ for an electron density of $n_e = 6 \times 10^{27}$ cm^{-3} (corresponding to $\rho = 2 \times 10^4$ g/cm^{-3} if $\mu_e = 2$), and for three different temperatures: $T = 2 \times 10^7$ K (black lines), $2{\times}10^6$ K (red lines) and $2{\times}10^5$ K (blue lines). The actual distributions, governed by quantum mechanics, are shown as solid lines while the Maxwell-Boltzmann distributions for the same n_e and T values are shown as dashed lines. The dotted line $n_{\rm max}$ is the maximum possible number distribution if all quantum states with momentum p are occupied. (right panel) Distributions in the limit $T = 0$, when all lowest available momenta are fully occupied. The blue line is for the same density as in the left panel, while the red line is for a density two times as high.

the distribution must equal n_e). The number of electrons with small values of p expected from classical physics, $n_{\rm MB}(p)$, then exceeds the maximum allowed by the Pauli exclusion principle, $n_{\rm max}(p)$. These electrons are forced to assume quantum states with higher p: the peak in the distribution $n_e(p)$ occurs at higher p. Due to the higher momenta and velocities of these electrons, the electron gas exerts a higher pressure than inferred from classical physics. This is called *degeneracy pressure*. If the temperature is decreased even more, e.g. at $T = 2 \times 10^5$ K (blue lines), the lowest momentum states become nearly all filled and $n_e(p)$ follows $n_{\rm max}(p)$ until it drops sharply. In this state of strong degeneracy, further decrease of T hardly changes the momentum distribution, so that the electron pressure becomes nearly *independent of temperature*.

Complete electron degeneracy

In the limit that $T \to 0$, all available momentum states are occupied up to a maximum value, while all higher states are empty, as illustrated in the right panel of Fig. 3.2. This is known as *complete degeneracy*, and the maximum momentum is called the *Fermi momentum* $p_{\rm F}$. Then we have

$$n_e(p) = \frac{8\pi p^2}{h^3} \qquad \text{for } p \le p_{\rm F}, \tag{3.33}$$

$$n_e(p) = 0 \qquad \text{for } p > p_{\rm F}. \tag{3.34}$$

The Fermi momentum is determined by the electron density through eq. (3.4), i.e. $\int_0^{p_{\rm F}} n_e(p)\,dp = n_e$, which yields

$$p_{\rm F} = h\left(\frac{3}{8\pi}n_e\right)^{1/3}. \tag{3.35}$$

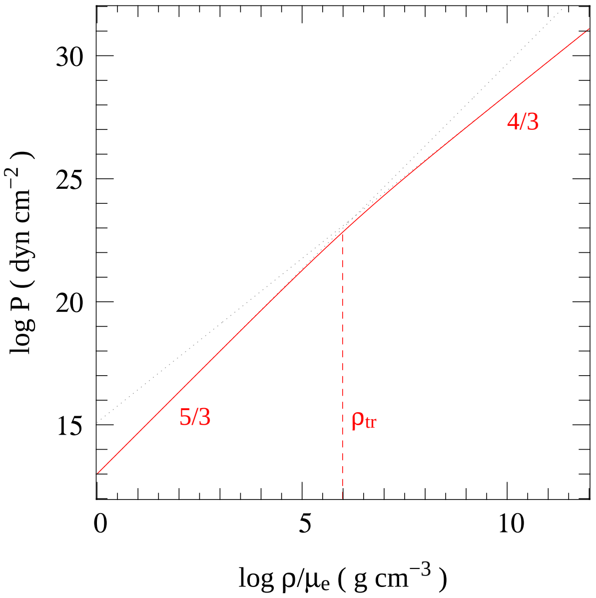


Figure 3.3. The equation of state for completely degenerate electrons. The slope of the $\log P$-$\log\rho$ relation changes from 5/3 at relatively low densities, where the electrons are non-relativistic, to 4/3 at high density when the electrons are extremely relativistic. The transition is smooth, but takes place at densities around $\rho_{\rm tr} \approx 10^6\mu_e\,{\rm g\,cm^{-3}}$.

The pressure of a completely degenerate electron gas is now easy to compute using the pressure integral eq. (3.6). It depends on whether the electrons are relativistic or not. In the *non-relativistic* limit we have $v = p/m$ and hence

$$P_e = \tfrac{1}{3}\int_0^{p_{\rm F}} \frac{8\pi p^4}{h^3 m_e}\,dp = \frac{8\pi}{15h^3 m_e}\,{p_{\rm F}}^5 = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3} {n_e}^{5/3}. \tag{3.36}$$

Using eq. (3.20) for n_e this can be written as

$$\boxed{P_e = K_{\rm NR}\left(\frac{\rho}{\mu_e}\right)^{5/3}} \tag{3.37}$$

where $K_{\rm NR} = \frac{h^2}{20m_e}(3/\pi)^{2/3}{m_{\rm u}}^{-5/3} = 1.00\times10^{13}$ [cgs units]. As more electrons are squeezed into the same volume, they have to occupy states with larger momenta, as illustrated in Fig. 3.2. Therefore the electron pressure increases with density, as expressed by eq. (3.37).

If the electron density is increased further, at some point the velocity of the most energetic electrons, $p_{\rm F}/m_e$, approaches the speed of light. We then have to replace $v = p/m$ by the relativistic kinematics relation (3.12). In the *extremely relativistic* limit when the majority of electrons move at relativistic speeds, we can take $v = c$ and

$$P_e = \tfrac{1}{3}\int_0^{p_{\rm F}} \frac{8\pi c p^3}{h^3}\,dp = \frac{8\pi c}{12h^3}\,p_{\rm F}^4 = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} {n_e}^{4/3}, \tag{3.38}$$

which gives

$$\boxed{P_e = K_{\rm ER}\left(\frac{\rho}{\mu_e}\right)^{4/3}} \tag{3.39}$$

with $K_{\rm ER} = \frac{hc}{8}(3/\pi)^{1/3}{m_{\rm u}}^{-4/3} = 1.24\times10^{15}$ [cgs units]. In the ER limit the pressure still increases with density, but with a smaller exponent ($\frac{4}{3}$ instead of $\frac{5}{3}$). The transition between the NR regime, eq. (3.37), and the ER regime, eq. (3.39), is smooth and can be expressed as a function of $x = p_{\rm F}/m_e c$, see K&W Chapter 15. Roughly, the transition occurs at a density $\rho_{\rm tr}$ given by the condition $p_{\rm F} \approx m_e c$, which can be expressed as

$$\rho_{\rm tr} \approx \mu_e\, m_{\rm u} \frac{8\pi}{3}\left(\frac{m_e c}{h}\right)^3. \tag{3.40}$$

The relation between P_e and ρ for a completely degenerate electron gas is shown in Fig. 3.3.

Partial degeneracy

Although the situation of complete degeneracy is only achieved at $T = 0$, it is a very good approximation whenever the degeneracy is strong, i.e. when the temperature is sufficiently low, as illustrated by Fig. 3.2. It corresponds to the situation when the degeneracy parameter $\psi \gg 0$ in eq. (3.31). In that case eqs. (3.37) and (3.39) can still be used to calculate the pressure to good approximation.

The transition between the classical ideal gas situation and a state of strong degeneracy occurs smoothly, and is known as *partial degeneracy*. To calculate the pressure the full expression eq. (3.31) has to be used in the pressure integral, which becomes rather complicated. The integral then depends on ψ, and can be expressed as one of the so-called *Fermi-Dirac* integrals, see K&W Chapter 15 for details (the other Fermi-Dirac integral relates to the internal energy density U). The situation of partial degeneracy corresponds to $\psi \sim 0$.

When $\psi \ll 0$ the classical description is recovered, i.e. eq. (3.31) becomes the Maxwell-Boltzmann distribution. In that case $1/(e^{(p^2/2m_ekT)-\psi} + 1) = e^{-(p^2/2m_ekT)+\psi}$ and therefore

$$\frac{2}{h^3}e^{\psi} = \frac{n_e}{(2\pi m_e kT)^{3/2}} \qquad \text{or} \qquad \psi = \ln \frac{h^3 n_e}{2(2\pi m_e kT)^{3/2}}.$$

This only holds for $\psi \ll 0$, but more generally it can be shown that $\psi = \psi(n_e/T^{2/3})$. We have to consider (partial) degeneracy if $\psi \gtrsim 0$, i.e. if

$$n_e \gtrsim \frac{2(2\pi m_e kT)^{3/2}}{h^3}. \tag{3.41}$$

The limit of strong (almost complete) degeneracy is reached when n_e is roughly a factor 10 higher.

Importance of electron degeneracy in stars

As a star, or its core, contracts the density may become so high that the electrons become degenerate and exert a (much) higher pressure than they would if they behaved classically. Since in the limit of strong degeneracy the pressure no longer depends on the temperature, this degeneracy pressure can hold the star up against gravity, regardless of the temperature. Therefore a degenerate star does not have to be hot to be in hydrostatic equilibrium, and it can remain in this state forever even when it cools down. This is the situation in *white dwarfs*.

The importance of relativity is that, when a degenerate star becomes more compact and the density increases further, the pressure increases less steeply with density. This has important consequences for massive white dwarfs, and we shall see that it implies that there is a maximum mass for which white dwarfs can exist (the Chandrasekhar mass).

We note that although electron degeneracy can be (very) important in stars, degeneracy of the *ions* is not. Since the ions have masses $\gtrsim 2000$ larger than electrons, their momenta ($p = \sqrt{2m\epsilon}$) are much larger at energy equipartition, and the condition (3.41) above (with m_e replaced by $m_{\rm ion}$) implies that much higher densities are required at a particular temperature. In practice this never occurs: before such densities are reached the protons in the atomic nuclei will capture free electrons, and the composition becomes one of (mostly) neutrons. Degeneracy of *neutrons* does become important when we consider neutron stars.

3.3.6 Radiation pressure

Photons can be treated as quantum-mechanical particles that carry momentum and therefore exert pressure when they interact with matter. In particular photons are *bosons* with $g_s = 2$ (two polarization states), so they can be described by the Bose-Einstein statistics, eq. (3.29). The number of photons is not conserved, they can be destroyed and created until thermodynamic equilibrium is achieved. This means that $\mu = 0$ in eq. (3.29) and hence

$$n(p)dp = \frac{2}{h^3}\frac{1}{e^{\epsilon_p/kT} - 1}4\pi p^2 dp \tag{3.42}$$

Photons are completely relativistic with $\epsilon_p = pc = h\nu$, so in terms of frequency ν their distribution in LTE becomes the *Planck function* for blackbody radiation:

$$n(\nu)d\nu = \frac{8\pi}{c^3}\frac{\nu^2 d\nu}{e^{h\nu/kT} - 1} \tag{3.43}$$

Applying eqs. (3.4) and (3.5) one can show that the photon number density and the energy density of radiation are

$$n_{\rm ph} = \int_0^\infty n(p)\,dp = b\,T^3 \tag{3.44}$$

$$U_{\rm rad} = \int_0^\infty pc\,n(p)\,dp = a\,T^4 \tag{3.45}$$

where $b = 20.3\,\mathrm{cm^{-3}\,K^{-3}}$ and a is the radiation constant

$$a = \frac{8\pi^5 k^4}{15h^3c^3} = 7.56 \times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}}.$$

Since photons are always extremely relativistic, $P = \frac{1}{3}U$ by eq. (3.14) and the *radiation pressure* is given by

$$\boxed{P_{\rm rad} = \tfrac{1}{3}aT^4} \tag{3.46}$$

Pressure of a mixture of gas and radiation

The pressure inside a star is the sum of the gas pressure and radiation pressure,

$$P = P_{\rm rad} + P_{\rm gas} = P_{\rm rad} + P_{\rm ion} + P_e.$$

where $P_{\rm rad}$ is given by eq. (3.46) and $P_{\rm ion}$ by eq. (3.19). In general P_e must be calculated as described in Sect. 3.3.5. In the classical limit it is given by eq. (3.21), and in the limits of non-relativistic and extremely relativistic degeneracy by eqs. (3.37) and (3.39), respectively. If the electrons are non-degenerate then the pressure can be written as

$$P = \tfrac{1}{3}aT^4 + \frac{\mathcal{R}}{\mu}\rho T.$$

If the electrons are strongly degenerate their pressure dominates over that of the (classical) ions, so in that case $P_{\rm ion}$ can be neglected in the total pressure.

The fraction of the pressure contributed by the gas is customarily expressed as β, i.e.

$$P_{\rm gas} = \beta P \qquad \text{and} \qquad P_{\rm rad} = (1-\beta)\,P. \tag{3.47}$$

3.3.7 Equation of state regimes

The different sources of pressure we have discussed so far dominate the equation of state at different temperatures and densities. In Fig. 3.4 the boundaries between these regimes are plotted schematically in the $\log T$, $\log\rho$ plane.

- The boundary between regions where radiation and ideal-gas pressure dominate is defined by $P_{\rm rad} = P_{\rm gas}$, giving $T/\rho^{1/3} = 3.2 \times 10^7 \mu^{-1/3}$ when T and ρ are expressed in cgs units. (Verify this by comparing eqs. 3.23 and 3.46.) This is a line with slope $\frac{1}{3}$ in the $\log T$ vs $\log\rho$ plane.
- Similarly, the boundary between the regions dominated by ideal-gas pressure and non-relativistic degenerate electron pressure can be defined by $P_{\rm gas,ideal} = P_{e,\rm NR}$ as given by eq. (3.37), giving $T/\rho^{2/3} = 1.21 \times 10^5 \mu\, \mu_e^{-5/3}$ (again with T and ρ in cgs units). This is a line with slope $\frac{2}{3}$ in the $\log T$-$\log\rho$ plane.
- The approximate boundary between non-relativistic and relativistic degeneracy is given by eq. (3.40), $\rho = 9.7 \times 10^5 \mu_e$ g/cm^3.
- At high densities the boundary between ideal gas pressure and extremely relativistic degeneracy is found by equating eqs. (3.23) and (3.39), giving $T/\rho^{1/3} = 1.50 \times 10^7 \mu\, \mu_e^{-4/3}$ (with T and ρ in cgs units), again a line with slope $\frac{1}{3}$.

As shown in Fig. 3.4, detailed models of zero-age (that is, homogeneous) main-sequence stars with masses between 0.1 and 100 $M_\odot$ cover the region where ideal-gas pressure dominates the equation of state. This justifies the assumptions made in Ch. 2 when discussing the virial theorem and its consequences for stars, and when estimating temperatures in the stellar interior.

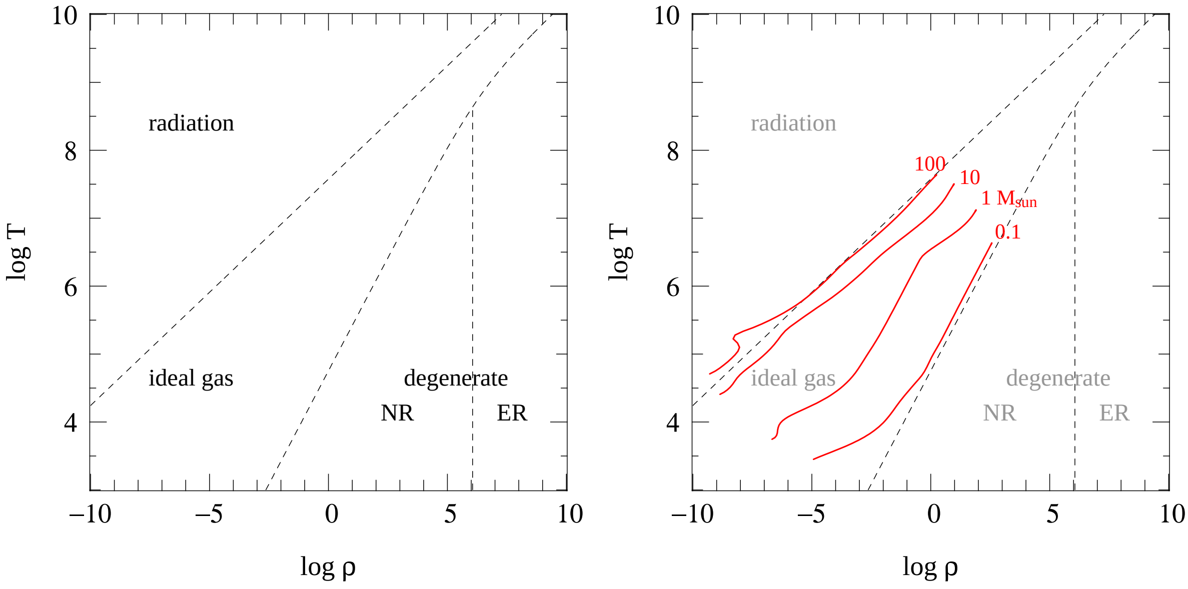


Figure 3.4. The equation of state for a gas of free particles in the $\log T$, $\log\rho$ plane. The dashed lines are approximate boundaries between regions where radiation pressure, ideal gas pressure, non-relativistic electron degeneracy and extremely relativistic electron degeneracy dominate, for a composition $X = 0.7$ and $Z = 0.02$. In the right panel, detailed structure models for homogeneous main-sequence stars of 0.1...100 $M_\odot$ have been added (solid lines). The 1 $M_\odot$ model is well within the ideal-gas region of the equation of state. In the 0.1 $M_\odot$ star electron degeneracy pressure is important, except in the outer layers (at low ρ and T). In stars more massive than 10 $M_\odot$, radiation pressure becomes important, and it dominates in the surface layers of the 100 $M_\odot$ model.

3.4 Adiabatic processes

It is often important to consider processes that occur on such a short (e.g. hydrodynamical) timescale that there is no heat exchange with the environment; such processes are *adiabatic*. To derive the properties of stellar interiors under adiabatic conditions we need several thermodynamic derivatives. We therefore start from the laws of thermodynamics.

The *first law* of thermodynamics (see Sec. 2.4) states that the amount of heat absorbed by a system is the sum of the change in its internal energy and the work done on the system. The *second law* of thermodynamics states that, for a reversible process, the change in entropy equals the change in the heat content divided by the temperature. Entropy is a state variable, unlike the heat content. For a unit mass (1 gram) of matter the combination of these laws can be expressed as

$$dq = T\,ds = du + P\,dv = du - \frac{P}{\rho^2}\,d\rho. \tag{3.48}$$

Here dq is the change in heat content, du is the change in internal energy ($u = U/\rho$ is the *specific* internal energy, i.e. per gram), s is the specific entropy (i.e. the entropy per unit mass) and $v = 1/\rho$ is the volume of a unit mass. Note that du and ds are exact differentials, whereas dq is not.

Differential form of the equation of state To compute general expressions for thermodynamic derivatives such as the specific heats and the adiabatic derivatives it is useful to write the equation of state in differential form, i.e.

$$\frac{dP}{P} = \chi_T \frac{dT}{T} + \chi_\rho \frac{d\rho}{\rho}, \tag{3.49}$$

where χ_T and χ_ρ are defined as

$$\chi_T = \left(\frac{\partial \log P}{\partial \log T}\right)_{\rho, X_i} = \frac{T}{P}\left(\frac{\partial P}{\partial T}\right)_{\rho, X_i}, \tag{3.50}$$

$$\chi_\rho = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{T, X_i} = \frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_{T, X_i}. \tag{3.51}$$

The subscript X_i means that the composition is held constant as well. In a general equation of state χ_T and χ_ρ can depend on T and ρ themselves, but if they are (approximately) constant then we can write the equation of state in power-law form:

$$P = P_0\,\rho^{\chi_\rho}\,T^{\chi_T}.$$

For example, for an ideal gas without radiation we have $\chi_T = \chi_\rho = 1$, while for a radiation-dominated gas $\chi_T = 4$ and $\chi_\rho = 0$.

3.4.1 Specific heats

The specific heats at constant volume c_V and at constant pressure c_P for a unit mass of gas follow from eq. (3.48):

$$c_V = \left(\frac{dq}{dT}\right)_v = \left(\frac{\partial u}{\partial T}\right)_v, \tag{3.52}$$

$$c_P = \left(\frac{dq}{dT}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P - \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P, \tag{3.53}$$

where a partial derivative taken at constant v is the same as one taken at constant ρ. For an ideal gas, with $u = U/\rho = \frac{3}{2}P/\rho$, we obtain from eq. (3.23) the familiar result $c_V = \frac{3}{2}\mathcal{R}/\mu$. For a radiation-dominated gas, eq. (3.45) yields $c_V = 4aT^3/\rho$. Using thermodynamic transformations and some algebraic manipulation (see Appendix), it follows quite generally that the specific heats are related by

$$c_P - c_V = \frac{P}{\rho T}\frac{{\chi_T}^2}{\chi_\rho}. \tag{3.54}$$

For an ideal gas this amounts to $c_P - c_V = \mathcal{R}/\mu$, and therefore $c_P = \frac{5}{2}\mathcal{R}/\mu$. For a radiation-dominated gas $\chi_\rho = 0$ and hence $c_P \to \infty$: indeed, since $P_{\rm rad}$ only depends on T, a change in temperature cannot be performed at constant pressure.

The ratio of specific heats is often denoted as γ:

$$\gamma = \frac{c_P}{c_V} = 1 + \frac{P}{\rho T c_V}\frac{{\chi_T}^2}{\chi_\rho}, \tag{3.55}$$

so that $\gamma = \frac{5}{3}$ for an ideal gas.

3.4.2 Adiabatic derivatives

The thermodynamic response of a system to adiabatic changes is measured by the so-called *adiabatic derivatives*. Two of these have special importance for stellar structure:

- The *adiabatic exponent*[2] $\gamma_{\rm ad}$ measures the response of the pressure to adiabatic compression or expansion, i.e. to a change in the density. It is defined as

  $$\gamma_{\rm ad} = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{\rm ad} \tag{3.56}$$

  where the subscript 'ad' means that the change is performed adiabatically, that is, at constant entropy. If $\gamma_{\rm ad}$ is constant then $P \propto \rho^{\gamma_{\rm ad}}$ for adiabatic changes. As we shall see later, $\gamma_{\rm ad}$ is related to the *dynamical stability* of stars.

- The *adiabatic temperature gradient* is defined as

  $$\nabla_{\rm ad} = \left(\frac{\partial \log T}{\partial \log P}\right)_{\rm ad} \tag{3.57}$$

  It is in fact another exponent that describes the behaviour of the temperature under adiabatic compression or expansion ($T \propto P^{\nabla_{\rm ad}}$ if $\nabla_{\rm ad}$ is constant), which turns out to be important for stability against *convection*.

---

[2] In many textbooks one finds instead the adiabatic exponents Γ_1, Γ_2, and Γ_3 introduced by Chandrasekhar. They are defined, and related to $\gamma_{\rm ad}$ and $\nabla_{\rm ad}$, as follows:

$$\Gamma_1 = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{\rm ad} = \gamma_{\rm ad}, \qquad \frac{\Gamma_2}{\Gamma_2 - 1} = \left(\frac{\partial \log P}{\partial \log T}\right)_{\rm ad} = \frac{1}{\nabla_{\rm ad}}, \qquad \Gamma_3 = \left(\frac{\partial \log T}{\partial \log \rho}\right)_{\rm ad} + 1.$$

They obey the relation

$$\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1}.$$

The adiabatic exponent For an adiabatic process $dq = 0$ in eq. (3.48) and therefore

$$du = \frac{P}{\rho^2}\, d\rho. \tag{3.58}$$

We have seen in Sect. 3.3.1 that for a perfect gas of free particles the internal energy density U is proportional to P, in both the NR and ER limits. For such a simple system we can therefore write, as we did in Sect. 2.3,

$$u = \phi\, \frac{P}{\rho} \tag{3.59}$$

with ϕ a constant (between $\frac{3}{2}$ and 3). If we differentiate this and substitute into eq. (3.58) we obtain for an adiabatic change

$$\frac{dP}{P} = \frac{\phi + 1}{\phi}\frac{d\rho}{\rho}. \tag{3.60}$$

Therefore, according to the definition of γ_{ad} (eq. 3.56),

$$\gamma_{\mathrm{ad}} = \frac{\phi + 1}{\phi} \qquad \text{(for a simple, perfect gas).} \tag{3.61}$$

- for *non-relativistic* particles (e.g. a classical ideal gas, NR degenerate electrons) $\phi = \frac{3}{2}$ and therefore $\gamma_{\mathrm{ad}} = \frac{5}{3}$
- for *extremely relativistic* particles (e.g. photons, ER degenerate electrons) $\phi = 3$ and therefore $\gamma_{\mathrm{ad}} = \frac{4}{3}$
- for a mixture of gas and radiation ($0 \leq \beta \leq 1$) and/or moderately relativistic degenerate electrons, $\frac{4}{3} \leq \gamma_{\mathrm{ad}} \leq \frac{5}{3}$

For a general equation of state, described by eq. (3.49), one can derive (see Appendix)

$$\gamma_{\mathrm{ad}} = \chi_\rho + \frac{P}{\rho T c_V}\chi_T{}^2. \tag{3.62}$$

Therefore γ_{ad} is related to the ratio of specific heats (eq. 3.55), $\gamma_{\mathrm{ad}} = \gamma\chi_\rho$. The γ's are equal if $\chi_\rho = 1$ (as in the case of an ideal gas).

The adiabatic temperature gradient By writing eq. (3.56) as $dP/P = \gamma_{\mathrm{ad}}\, d\rho/\rho$ for an adiabatic change, and eliminating $d\rho$ with the help of eq. (3.49), we obtain a general relation between the adiabatic temperature gradient ∇_{ad} and the adiabatic exponent γ_{ad}:

$$\nabla_{\mathrm{ad}} = \frac{\gamma_{\mathrm{ad}} - \chi_\rho}{\gamma_{\mathrm{ad}}\chi_T}, \tag{3.63}$$

This gives the following limiting cases:

- for an ideal gas without radiation ($\beta = 1$) we have $\chi_T = \chi_\rho = 1$, which together with $\gamma_{\mathrm{ad}} = \frac{5}{3}$ gives $\nabla_{\mathrm{ad}} = \frac{2}{5} = 0.4$.
- for a radiation-dominated gas ($\beta = 0$) $\chi_T = 4$ and $\chi_\rho = 0$ so that $\nabla_{\mathrm{ad}} = \frac{1}{4} = 0.25$.

For a general equation of state one has to consider the general expression for $\gamma_{\rm ad}$ (eq. 3.62) in eq. (3.63). One can also derive the following relation for $\nabla_{\rm ad}$ (see Appendix):

$$\nabla_{\rm ad} = \frac{P}{\rho T c_P} \frac{\chi_T}{\chi_\rho}. \tag{3.64}$$

We give some important results without derivations, which can be found in K&W Chapters 13.2 and 16.3 or in Hansen Chapter 3.7:

- for a mixture of gas and radiation with $0 < \beta < 1$, $\nabla_{\rm ad}$ and $\gamma_{\rm ad}$ both depend on β and take on intermediate values, i.e. $0.25 < \nabla_{\rm ad} < 0.4$.
- for a non-relativistic degenerate gas, we have to consider that although electrons dominate the pressure, there is a (tiny) temperature dependence due to the ion gas which must be taken into account in calculating χ_T and therefore $\nabla_{\rm ad}$. After some manipulation it can be shown that in this case $\nabla_{\rm ad} = 0.4$, as for the ideal classical gas.
- for an extremely relativistic degenerate gas one also has to consider that while the electrons are relativistic, the ions are still non-relativistic. It turns out that in this limit $\nabla_{\rm ad} = 0.5$.

3.5 Ionization

We have so far implicitly assumed complete ionization of the gas, i.e. that it consists of bare atomic nuclei and free electrons. This is a good approximation in hot stellar interiors, where $T > 10^6$ K so that typical energies kT are much larger than the energy needed to ionize an atom, i.e. to knock off a bound electron. In the cooler outer layers of a star, however, we need to consider the *partial ionization* of the elements. In this case quasi-static changes of the state variables (ρ and T) will lead to changes in the degree of ionization. This can have a large effect on the thermodynamic properties of the gas, e.g. on $\gamma_{\rm ad}$ and $\nabla_{\rm ad}$.

In LTE the number densities of ionized and neutral species are determined by the *Saha equation*

$$\frac{n_{r+1}}{n_r} n_e = \frac{u_{r+1}}{u_r} \frac{2(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT} \tag{3.65}$$

where n_r and n_{r+1} indicate the number densities of r and $r+1$ times ionized nuclei, χ_r is the ionization potential, i.e. the energy required to remove the r-th bound electron, and u_r and u_{r+1} are the partition functions. The partition functions depend on T but can in most cases be approximated by the statistical weights of the ground states of the bound species. (This equation can be derived from statistical mechanics, e.g. see K&W Chapter 14.1.)

3.5.1 Ionization of hydrogen

As an example, we consider the simple case where the gas consists only of hydrogen. Then there are just three types of particle, electrons and neutral and ionized hydrogen, with $u_{\rm H} = u_0 = 2$ and $u_{\rm H+} = u_1 = 1$. We write their number densities as n_+ and n_0 so that

$$\frac{n_+}{n_0} n_e = \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{\rm H}/kT} \tag{3.66}$$

where $\chi_{\rm H} = 13.6$ eV. The gas pressure is given by $P_{\rm gas} = (n_0 + n_+ + n_e)\, kT$ and the density is $\rho = (n_0 + n_+)\, m_u$. The *degree of ionization* is defined as

$$x = \frac{n_+}{n_0 + n_+} \tag{3.67}$$

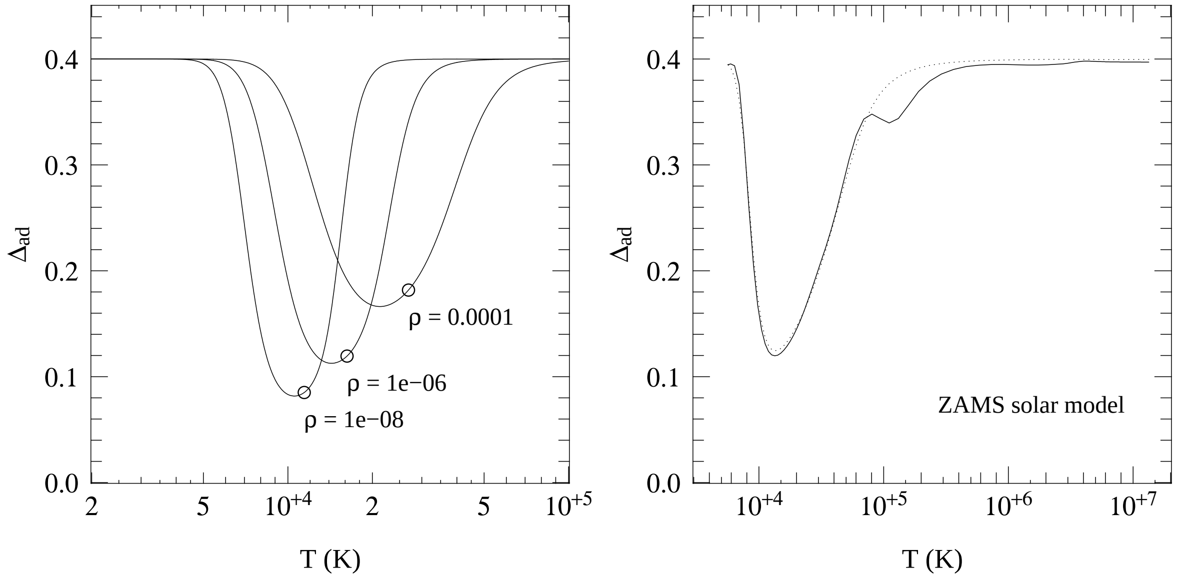


Figure 3.5. The adiabatic temperature gradient $\nabla_{\rm ad}$ plotted against temperature. The left panel shows the effect of partial ionization for the simple case of a pure hydrogen gas, for three values of the density (10^{-4}, 10^{-6} and 10^{-8} g/cm^3). When hydrogen is partially ionized, $\nabla_{\rm ad}$ is decreased below its ideal-gas value of 0.4. The circles indicate the points where the degree of ionization $x = 0.5$, close to the minimum of $\nabla_{\rm ad}$. As the density increases, a higher temperature is needed to reach the same ionization degree. The right panel shows how $\nabla_{\rm ad}$ varies with temperature in a detailed stellar model of 1 $M_\odot$, between the surface (at $T \approx 6000$ K) and the centre (at $T \approx 1.5 \times 10^7$ K). Apart from the hydrogen ionization zone around 10^4 K, a second depression of $\nabla_{\rm ad}$ around 10^5 K is seen which is due to the first ^{4}He ionization zone. The second He ionization zone is merged with H ionization because it occurs at similar temperatures and densities. Note that the region where $T < 10^6$ K comprises only the outer 1 % of the mass of the Sun. (The dotted line shows how $\nabla_{\rm ad}$ would vary with T in this model if the composition were pure hydrogen, as was assumed in the left panel.)

so that $P_{\rm gas}$ can be written in terms of the degree of ionization

$$P_{\rm gas} = (1 + x)\mathcal{R}\rho T \tag{3.68}$$

We can then rewrite Saha's equation as

$$\frac{x^2}{1 - x^2} = \frac{(2\pi m_e)^{3/2}}{h^3} \frac{(kT)^{5/2}}{P_{\rm gas}} e^{-\chi_{\rm H}/kT} \tag{3.69}$$

We see that the degree of ionization increases with T, as expected since more atoms are broken up by the energetic photons. However, x decreases with gas pressure (or density) when T is kept constant, because this increases the probability of recombination which is proportional to n_e. From eq. (3.68) we see that the mean molecular weight $\mu = 1/(1 + x)$ decreases as hydrogen becomes ionized (one atomic mass is divided over two particles).

To estimate the effect on the thermodynamic properties of the gas, we note that in the case of partial ionization the internal energy has a contribution from the available potential energy of recombination. Per unit volume this contribution is equal to $n_+\chi_{\rm H}$, so per unit mass it equals $n_+\chi_{\rm H}/\rho = x\chi_{\rm H}/m_{\rm u}$. Thus

$$u = \frac{3}{2}\frac{P_{\rm gas}}{\rho} + x\frac{\chi_{\rm H}}{m_{\rm u}} = \tfrac{3}{2}(1 + x)\mathcal{R}T + x\frac{\chi_{\rm H}}{m_{\rm u}}. \tag{3.70}$$

A small increase in temperature increases the degree of ionization, which results in a large amount of energy being absorbed by the gas. In other words, the *specific heat* of a partially ionized gas will be much larger than for an unionized gas, or for a completely ionized gas (in the latter case $x = 1$ so that the second term in eq. (3.70) becomes a constant and therefore irrelevant).

Now consider what happens if the gas is adiabatically compressed. Starting from neutral hydrogen, for which $\nabla_{\rm ad} = 0.4$, the temperature initially increases as $T \propto P^{0.4}$. Further compression (work done on the gas) increases u, but when partial ionization sets in most of this energy goes into raising the degree of ionization (second term of eq. 3.70) and only little into raising the temperature (first term). In other words, T increases less strongly with with P, and therefore $\nabla_{\rm ad} < 0.4$. A detailed calculation (e.g. see K&W Chapter 14.3) shows that under typical conditions $\nabla_{\rm ad}$ reaches a minimum value of ≈ 0.1 when $x \approx 0.5$. As the gas becomes almost fully ionized, $\nabla_{\rm ad}$ rises back to 0.4. The variation of $\nabla_{\rm ad}$ with temperature for a pure hydrogen gas is shown in the left panel of Fig. 3.5 for different values of the density.

The decrease of $\nabla_{\rm ad}$ in partial ionization zones can induce *convection* in the outer layers of stars, as we shall see in Ch. 4. Similarly it can be shown that $\gamma_{\rm ad}$ decreases in partial ionization zones, from $\frac{5}{3}$ to $\gamma_{\rm ad} \approx 1.2$ when $x \approx 0.5$. This has consequences for the stability of stars, as we shall also see.

3.5.2 Ionization of a mixture of gases

In a mixture of gases the situation becomes more complicated because many, partly ionized species have to be considered, the densities of which all depend on each other (see e.g. K&W Chapter 14.4-14.5). However the basic physics remains the same as considered above for the simple case of pure hydrogen. The effect on the thermodynamic properties is that e.g. $\nabla_{\rm ad}$ can show additional deviations below 0.4 at different temperatures, especially where helium (the second-most abundant element in stars) is partially ionized. This is illustrated in Fig. 3.5b which shows the variation of $\nabla_{\rm ad}$ with temperature in a homogeneous model for the initial Sun.

3.5.3 Pressure ionization

As ρ increases indefinitely, the Saha equation gives $x \to 0$, i.e. ionized gas recombines to form atoms. This is obviously nonsense at very high density, and becomes incorrect when the average distance between ions becomes less than an atomic radius. In this situation the ionization energy is suppressed (there are fewer bound excited states; see Fig. 3.5.3), a situation known as *pressure ionization*.

Consider the case of hydrogen: the volume per H atom is $1/n_{\rm H}$ so that $\langle d \rangle = (\frac{4\pi}{3} n_{\rm H})^{-1/3}$. Pressure ionization sets in when $\langle d \rangle \lesssim a_0 = 5 \times 10^{-9}$ cm (the Bohr radius). This implies

$$n_{\rm H} \gtrsim \frac{1}{\frac{4\pi}{3} a_0{}^3}$$

or $\rho = n_{\rm H} m_{\rm H} \gtrsim 3\,{\rm g\,cm^{-3}}$. Other elements are pressure-ionized at similar values of the density, within an order of magnitude. At densities $\gtrsim 10\,{\rm g\,cm^{-3}}$, therefore, we can again assume complete ionization.

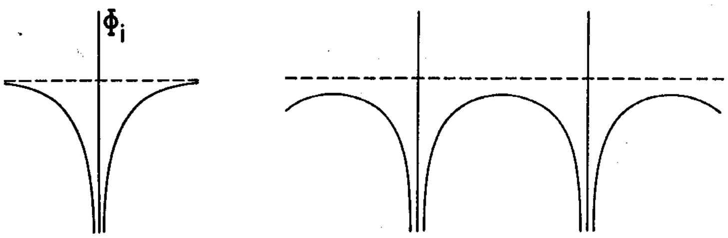


Figure 3.6. Schematic depiction of the electrostatic potential of an isolated ion (left) and the superposition of the potentials of neighbouring ions (right). Figure reproduced from Kippenhahn & Weigert.

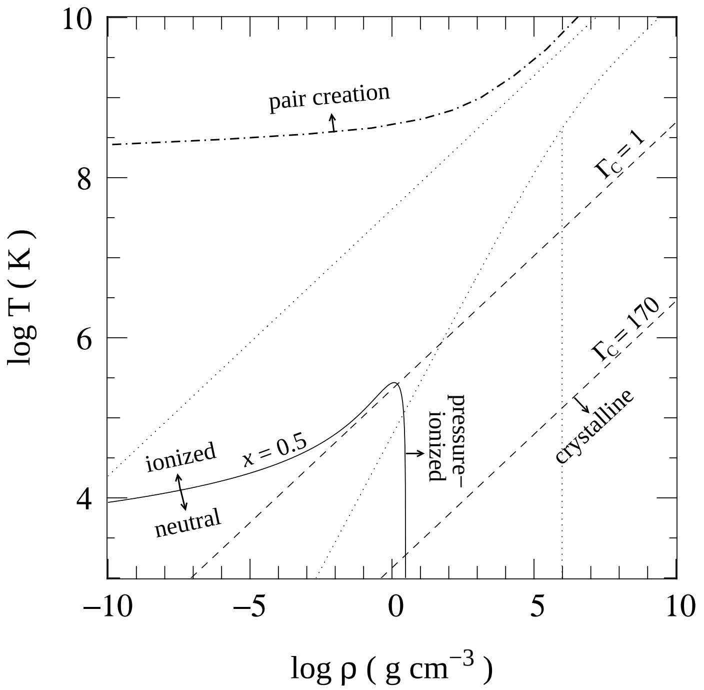


Figure 3.7. The equation of state in the ρ, T plane for a pure hydrogen gas. The dotted lines are the borders, also shown in Fig. 3.4, between regions where radiation, ideal gas and degenerate electrons dominate the pressure. The solid line shows where the ionization fraction of hydrogen is 0.5 according to the Saha equation, and where hydrogen becomes pressure-ionized at high density. The dashed lines show where the Coulomb interaction parameter $\Gamma_{\rm C}$ equals 1, above which Coulomb interactions become important, and where $\Gamma_{\rm C} = 170$, above which the ions form a crystralline lattice. Above the dash-dotted line e^+e^- pairs play an important role in stellar interiors.

Fig. 3.7 shows the approximate boundary in the density-temperature diagram between neutral and ionized hydrogen according the Saha equation for $\rho < 1\,{\rm g\,cm^{-3}}$, and as a result of pressure ionization at higher densities.

3.6 Other effects on the equation of state

3.6.1 Coulomb interactions and crystallization

In Sect. 3.2.1 we showed that Coulomb interactions normally have very small energies compared to the kinetic energies of the ions and electrons. The ratio of Coulomb energy to kinetic energy is usually called the Coulomb parameter $\Gamma_{\rm C}$, defined as in eq. (3.2):

$$\Gamma_{\rm C} = \frac{Z^2 e^2}{d\,kT} = \frac{Z^2 e^2}{kT}\left(\frac{4\pi\rho}{3Am_{\rm u}}\right)^{1/3} = 2.275\times10^5\,\frac{Z^2}{A^{1/3}}\,\frac{\rho^{1/3}}{T}, \tag{3.71}$$

where in the last equality the numerical factor is in cgs units. Coulomb interactions start to become important in stellar interiors when $\Gamma_{\rm C} \gtrsim 1$. In Fig. 3.7 the location of the condition $\Gamma_{\rm C} = 1$ is indicated in the ρ-T diagram. Comparing with Fig. 3.4 shows that the effect of Coulomb interactions is significant in very low-mass stars.

If $\Gamma_{\rm C} \gg 1$ the thermal motions of the ions are overwhelmed by the Coulomb interactions. In this situation the ions will tend to settle down into a conglomerate with a lower energy, in other words they will form a crystalline lattice. Detailed estimates indicate that this transition takes place at a critical value of $\Gamma_{\rm C} \approx 170$. This condition is also indicated in Fig. 3.4 for a pure hydrogen gas. In reality, this situation will never occur in hydrogen-rich stellar interiors, but it can take place in cooling white dwarfs (in which the temperature gradually decreases with time while the density remains constant). White dwarfs are usually composed of carbon and oxygen, so in this case we have to take into account the composition which raises the temperature at which the transition occurs (the ‘melting’ temperature) by a factor $Z^2/A^{1/3}$ according to eq. (3.71).

Finally we note that crystallization only occurs in the region where the electrons are strongly degenerate. You may verify that the Coulomb interaction energy between electrons and ions (Ze^2/d) is always smaller than the typical electron energy ($p_{\rm F}^2/2m_e$). The electrons therefore behave like a free degenerate gas, even if the ions form a crystalline structure.

3.6.2 Pair production

A very different process can take place at very high temperatures and relatively low densities. A photon may turn into an electron-positron pair if its energy $h\nu$ exceeds the rest-mass energy of the pair, $h\nu > 2m_ec^2$. This must take place during the interaction with a nucleus, since otherwise momentum and energy cannot both be conserved. Pair production takes place at a typical temperature $kT \approx h\nu \approx 2m_ec^2$, or $T \approx 1.2\times10^{10}$ K. However, even at $T \sim 10^9$ K the number of energetic photons in the tail of the Planck distribution (eq. 3.43) is large enough to produce a large number of $\mathrm{e^+e^-}$ pairs. The newly created positrons tend to be annihilated quickly by the inverse reaction ($\mathrm{e^+} + \mathrm{e^-} \rightarrow \gamma$), as a result of which the number of positrons reaches equilibrium. At a few times 10^9 K, depending on the electron density, the number of positrons is a significant fraction of the number of electrons.

Pair production is similar to an ionization process: an increase in temperature leads to an increase in the number of particles at the expense of the photon energy (and pressure). Therefore pair production gives rise to a decrease of the adiabatic gradient $\gamma_{\rm ad}$ and of $\nabla_{\rm ad}$, similar to partial ionization. This is the main importance of pair production for stellar evolution: it affects the stability of very massive stars in advanced stages of evolution (when their temperature may reach values in excess of 10^9 K) and can trigger their collapse.

Suggestion for further reading

The contents of this chapter are also covered by Chapters 13 to 16 of KIPPENHAHN & WEIGERT. However, a more elegant derivation of the equation of state, which is also more consistent with the way it is derived in these lecture notes, is given in Chapter 3 of HANSEN, KAWALER & TRIMBLE (*Stellar Interiors*, Springer-Verlag, 2004). Explicit expressions for many of the results that are only mentioned here can be found in this book.