Structure and Evolution of Stars.

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Outline

1. **I. Basic Physical Ingredients (these notes).**
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   - The virial theorem
   - Evolutionary time scales: τ_{ff}, τ_{KR} and τ_{nuc}
   - The Jeans' mass
   - Stability with polytropes
   - Degrees of freedom
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Polytropes: a simple introduction

The thermodynamic concept of "polytropic transformations" (Chandrasekhar 1958) is a crucial one in physics. For a stirred perfect gas, this concept is used when, in the stirring, $dQ = cdT$, T being the temperature in ^{o}K and c a constant *(specific heat)*. If the energy is preserved, we know that the equation of the adiabatic transformations holds:

$$P \propto \rho^{\gamma} \tag{1}$$

where ρ is the density, P is the pressure and γ is the adiabatic exponent ($\equiv 5/3$ for a perfect gas with 3 degrees of freedom). More generally, we can show that in polytropes a relation:

$$P \propto \rho^{\delta} \tag{1bis}$$

remains valid, even when $\delta \neq \gamma$. For $c = 0, \delta = \gamma$ the process is adiabatic, for $\delta = 1$ it's isothermal.

Indeed, if you consider the equation of state of a perfect gas $PV = nRT$, where $R = kN_A = 8.3144621$ J mol^{-1} K^{-1} is the ideal gas constant, n is the number of moles and V is the volume, you have:

$$P = \frac{n(kN_A)T}{V}$$

Here $k = 1.3806 \cdot 10^{-23}$ J/K is the Boltzmann's constant and $N_A = 6.022 \cdot 10^{23}$ is the Avogadro's number. In a unit volume the mass is ρ and nN_A/V is the total number of particles in V. Then:

$$P = \frac{k}{\mu m_H}\rho T. \qquad (2)$$

Here μ is the *molecular weight* and m_H is the unit atomic mass. For constant T, the pressure P is proportional to ρ.

Polytropes, continued

Now, let's have a more general transformation with specific heat *c*. Since we know that:

$$c_V = \left(\frac{dQ}{dT}\right)_V, \quad c_P = \left(\frac{dQ}{dT}\right)_P, \quad R = c_P - c_V, \quad \gamma = \frac{c_P}{c_V}$$

we can also write:

$$c = \frac{dQ}{dT} \quad (\textit{for the given transformation})$$

For the first principle of Thermodynamics (with $dQ = cdT$):

$$c_V dT = cdT - PdV$$

From the equation of state (1 mole):

$$pV = RT, \quad PdV + VdP = RdT$$

Polytropes, continued

Hence, with some algebra:

$$\frac{c_V - c}{R} PdV + \frac{c_V - c}{R} VdP - VdP = 0$$

or

$$\frac{c_V - c}{c_P - c_V} PdV + \frac{c_V - c}{c_P - c_V} VdP - VdP = 0$$

This has the form:

$$aPdV + (a-1)VdP = 0 \quad \Rightarrow \frac{a}{a-1} PdV + VdP = 0$$

Now we can put δ = $a/(a-1)$ and multiply by $V^{(\delta-1)}$ to get:

$$d(PV^{\delta}) = 0, \quad PV^{\delta} = cost, \quad or\ P \propto \rho^{\delta}$$

Adiabatic exponents in presence of radiation

The polytropic exponent δ, or the adiabatic one γ are defined by relations like:

$$P \propto \rho^{\gamma}; \quad T \propto \rho^{\gamma-1} etc...$$

Differentiating them one has:

$$\frac{dT}{T} = (\gamma - 1)\frac{d\rho}{\rho}; \quad \frac{dP}{P} = \gamma\frac{d\rho}{\rho}$$

This holds for a perfect gas. What happens if the pressure is due also to radiation? The gas pressure will represent a fraction β and the radiation pressure a fraction $(1-\beta)$ of the total. It can be shown that, in this case, for an adiabatic transformation one can still derive similar relations, but the exponents are no longer identically the same as γ for a pure gas (Chandrasekhar 1958, Chapter IV).

Adiabatic exponents in presence of radiation: continued

In particular:

$$\frac{dP}{P} = \Gamma_1 \frac{d\rho}{\rho}; \quad \frac{dP}{P} = \frac{\Gamma_2}{\Gamma_2 - 1} \frac{dT}{T}; \quad \frac{dT}{T} = (\Gamma_3 - 1) \frac{d\rho}{\rho}.$$

Hence:

$$\frac{dlnP}{dln\rho} = \Gamma_1; \quad \frac{dlnP}{dlnT} = \frac{\Gamma_2}{\Gamma_2 - 1}; \quad \frac{dlnT}{dln\rho} = \Gamma_3 - 1 \tag{3}$$

It can also be shown that when $\beta < 1$, also the molecular weight is variable. One then defines:

$$\nabla_\mu = \frac{dlnT}{dln\mu} \tag{3'}$$

These relations will be useful in defining the borders of the zones where radiation or convection dominate.

Virial systems

Let's consider an interstellar cloud made of many particles of masses m_i and a reference frame where the distance vector of m_i from the origin is $\vec{r}_i$. Each particle is subject only to internal forces, namely gravity. Such a system is called a "virial" system (Clausius 1870). For it:

$$\Sigma m_i \frac{d^2 \vec{r}_i}{dt^2} = -G\Sigma \frac{M m_i}{r_i^3} \vec{r}_i$$

Consider now the scalar product by $\vec{r}_i$:

$$\Sigma m_i \vec{r}_i \ddot{\vec{r}}_i = -G\Sigma \frac{M m_i}{r_i} \equiv \Omega$$

where Ω is the potential gravitational energy. Let's now use the identity:

$$\frac{1}{2}\frac{d^2(mr^2)}{dt^2} = m\dot{r}^2 + mr\ddot{r} \quad (4)$$

The virial theorem: equilibrium

The term $\Sigma(mr_i^2)$ is the moment of inertia of the cloud, I. Hence

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2E_K + \Omega$$

(E_K = total kinetic energy, equal to the thermal energy. Then $E_K = U \equiv \frac{3}{2}N_{part.}kT$, or $2E_K = 3\int PdV$). When the cloud reaches equilibrium:

$$2U + \Omega = 0 \quad or\ U = -\frac{1}{2}\Omega \quad or\ \Omega = -3\int PdV \qquad (5)$$

This fact is in itself quite remarkable! Indeed, if you call E the total energy ($E = U + \Omega$), then you have:

$$E = -\frac{1}{2}\Omega + \Omega = U - 2U \Rightarrow E = -U \qquad (6)$$

Increasing E results in a *decrease* in U (and T): self-gravitating clouds (stars) have a NEGATIVE specific heat.

The free-fall time scale

For a nebula in a state of motion under its gravity, one has:

$$\rho \frac{\partial^2 r}{\partial t^2} + \frac{\partial P}{\partial r} = -G \frac{M_r \rho}{r^2} \qquad (7)$$

Initially, no pressure is perceived ($|dP/dr| \sim 0$). Then suppose that the cloud collapses so that the velocity is proportional to the distance from the centre ("homologous" collapse). Introducing in eq. (7):

$$M_r \equiv (M_r)_0 = \frac{4}{3}\pi \rho_0 r_0^3$$

it can be shown (next slide) that the time required for the collapse (*free fall time scale*) is:

$$\tau_{ff} = \left(\frac{3\pi}{32 G \rho_0}\right)^{1/2} \quad or\ \tau_{ff} \sim \frac{1}{\sqrt{G\rho}}\ within\ a\ factor\ 2 \qquad (8)$$

Free-fall time: demonstration

Eq. (7) can be rewritten, for a homologous collapse, after multiplication by the velocity:

$$\frac{dr}{dt}\frac{d^2r}{dt^2} = -\frac{4}{3}\pi G\rho_0 r_0^3 \frac{1}{r^2}\frac{dr}{dt} \rightarrow \frac{1}{2}\left(\frac{dr}{dt}\right)^2 = -\frac{4}{3}\pi G\rho_0 \frac{r_0^3}{r} + C$$

If at t=0 the cloud had no movement, $C = 4/3\pi G\rho_0 r_0^2$. Hence, substituting $\theta = r/r_0$, $K = (8/3\pi G\rho_0)^{1/2}$:

$$\frac{d\theta}{dt} = -K\left(\frac{1}{\theta} - 1\right)^{1/2} \Rightarrow \theta = cos^2\chi$$

$$cos^2\chi\frac{d\chi}{dt} = \frac{K}{2} \rightarrow (cos2\chi + 1)\frac{d\chi}{dt} = K$$

$$\chi + \frac{1}{2}sin2\chi = Kt + C_1 \rightarrow t_{ff} = \frac{\pi}{2K}$$

(at $t = 0$, $r = r_0$ and $C_1 = 0$; at $r = 0$, $\chi = \pi/2$).

The Kelvin-Helmoltz time scale

In general, if E is constant, only half the gravitational energy goes into keeping the gas warm. You may do what you want of the rest (e.g. breaking dust particles and molecules; if the gas has no "non-thermal" degrees of freedom, half of the garvitational energy can be radiated).

It is easy to show that, if R is the stellar radius and M_* is the total mass:

$$\Omega = -\int_0^{M_*} G\frac{MdM}{r(M)} = -\frac{3}{5}G\frac{M_*^2}{R} \tag{9}$$

The energy that can be radiated is half that value. If it is emitted at the present solar rate (3.8×10^{33} erg/sec), then the *Kelvin-Helmoltz* time scale is:

$$\tau_{KH} = \frac{3GM_\odot^2}{10R_\odot L_\odot} \simeq 2 \times 10^7 \; yr \tag{10}$$

In 1920 Rutherford and Holmes published the results of several years of measurements on ancient rocks. From the abundance of actinides they derived for the Earth an age larger than 1 Gyr. That was the proof that gravity cannot be the main source of energy for the Sun: it would be much younger than the Earth itself.
After the discovery of Helium, sir Eddington (well before Quantum Mechanics was formalized by Schroedinger & Heisenberg) noticed that a sufficiently long time scale for the Sun would be guaranteed by the fusion of 4 protons into a He nucleus.
The binding energy of He is 26.71 MeV (1 MeV = 1.6×10^{-13} J =1.6×10^{-7} erg). This corresponds to 0.7 % of the mass of 4 protons.

continued

Let's assume the Sun burns about 10% of its H mass (which is about correct).
How long does this take?
A mass m corresponds to an energy mc^2, hence:

$$\Delta E = 0.1 \times 1.989 \cdot 10^{33} x 0.007 \times (2.99 \cdot 10^{10})^2 \simeq 1.3 \times 10^{51} erg$$

Suppose the Sun radiates at its present power, $L_\odot = 3.8 \cdot 10^{33}$erg s^{-1}. One has a time scale:

$$\tau_{nucl} = \frac{1.3 \cdot 10^{51}}{3.8 \cdot 10^{33}} = 3.4 \cdot 10^{17} s \Rightarrow > 10^{10} yr$$

It works! Five years later Gamow (1928) showed that a simple exercise in solving Schroedinger's equation could allow the Sun to do that (Tunnel Effect). Only in 1938 Bethe and Critchfield published a detailed list of reactions that perform the job.

Equilibrium or Collapse? The Jeans' mass

There is another problem with the virial theorem applied to the conditions of the thinnest and coldest phases of the interstellar medium (ISM). Let's assume equilibrium, using proper expressions for Ω and U:

$$\Omega = -\frac{3GM^2}{5R}, \quad U = \frac{3M}{2\mu m_H}kT, \quad 2U + \Omega = 0$$

Since $M(r) = 4/3\pi r^3 \rho$ and $\rho = n\mu m_H$, you get:

The Jeans' Mass: continued

$$M_J = \left(\frac{5k}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\mu m_H}\right)^{1/2}\frac{T^{3/2}}{n^{1/2}} \simeq 2\times 10^{35}\frac{T^{3/2}}{n^{1/2}} \tag{11}$$

If T=10K, n=10cm^{-3}, M$_J \simeq 2\times 10^{35}$g, or 1000 $M_\odot$. This is called the "Jeans' mass" and compares well with the masses of "open clusters" in the Galaxy, not of stars. Individual stars form in comapct cores of ISM clouds, with $T\sim 20-30$K and $n \gtrsim 10^8$cm^{-3}. The Jeans' mass then is 20-30 $M_\odot$. Smaller masses form by fragmentation. Cloud cores form by a slow accumulation, induced by extra pressures (Galactic dynamics, magnetic fields). In this case:

$$2U + \Omega = 3\int PdV \tag{5bis}$$

The last term is the external work added.

Conditions for equilibrium

Suppose a nebula contracts till equilibrium is reached. Then in eq. (7) the pressure gradient equals the gravitational pull:

$$\frac{dP}{dr} = -\rho g = -G\rho\frac{M(r)}{r^2} \tag{12}$$

Let's put:

$$dM = 4\pi\rho r^2 dr, \quad r = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$$

Then:

$$dP = -G\frac{4\pi}{3r^2}\rho^2 r^3 dr = -G\rho^2 r\frac{4\pi dM}{3\cdot 4\pi\rho r^2}$$

yielding:

$$P \propto \rho^{4/3} \tag{13}$$

This is the weight of matter. Any stable thermal pressure must behave as a polytrope of exponent $\delta \geq 4/3$.

Degrees of freedom

In statistical mechanics, if we have q degrees of freedom, then

$$\gamma = \frac{C_P}{C_V} = 1 + \frac{2}{q} \tag{14}$$

If $q = 3$, $\gamma = 5/3$, if $q = 6$, $\gamma = 4/3$. Hence for $q \geq 6$ the structure is not stable.
Similar concepts are valid for polytropic structures.
Furthermore:

$$\gamma \Rightarrow \delta = \frac{n+1}{n}, \tag{14$'$}$$

where n is the polytropic *index*:

$$n \rightarrow 3/2, \delta \rightarrow 5/3; n \rightarrow 3, \delta \rightarrow 4/3.$$

Radiative transport

When energy is transported by the interaction of photons with matter (photons being absorbed and re-emitted by each stellar layer, with a very short mean free path) the fraction of the flux dF_ν across a layer of depth dr will be proportional to the flux itself F_ν; the coefficient $\bar{k}_\nu$ describes the interaction between radiation and matter per unit volume. Using the coefficient per unit *mass* k_ν ($\bar{k}_\nu = k_\nu \rho$):

$$dF_\nu = -F_\nu k_\nu \rho dr$$

Hence:

$$\frac{dF_\nu}{dr} = -k_\nu \rho F_\nu \tag{15}$$

And since $dP_r = dF_r/c$,

$$\frac{dP_r}{dr} = -\frac{k_\nu \rho L(r)}{4\pi c r^2} \tag{15'}$$

The Molecular weight

In a unit volume (where the mass is ρ) the number of particles is equal to the mass of the gas divided by the average mass of each particle.:

$$n = \frac{\rho}{\mu m_H}$$

Here μ is called *molecular weight* and m_H is the unit atomic mass. In a stellar plasma (with ionized gases) the particles include free electrons. How many particles do you have? Suppose the density ρ is made of a percentage X of hydrogem, a percentage Y of He, and Z of heavier elements ($A > 4$). At full ionization, every hydrogen atom releases one electron, hence the particles associated to H are:

$$n_H = 2X\frac{\rho}{m_H}$$

The molecular weight: continued

Similarly, for He we have a ion of mass 4 m_H and 2 electrons. The mass is 4 (electrons can be neglected) but the number of particles is 3!

$$n_{He} = \frac{3}{4} Y \frac{\rho}{m_H}$$

For heavier elements (total abundance in the Sun $Z = 0.015$) the most abundant are those made of α particles (^{12}C, ^{16}O, ^{20}Ne...), for which the charge is one half the atomic mass ($A/2$). Hence a nucleus of mass A_i releases $Z_i \simeq A_i/2$ electrons (plus one ion). If $\bar{A}$ is an average mass value:

$$n_{A>4} = \frac{\bar{A}}{2} \frac{1}{\bar{A}} Z \frac{\rho}{m_H}$$

The molecular weight: continued

This results in:

$$\mu = \left[\frac{m_H}{\rho}\frac{\rho}{m_H}\right]\frac{1}{\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]} \tag{16}$$

For pure hydrogen (X =1) μ equals 0.5; for a mixture of pure heavy elements [e.g. in the innermost cores of massive stars before a Core-Collapse Supernova (CCSN) occurs], $\mu = 2$. For a ionized gas of the solar composition ($X = 0.71$; $Y = 0.275$; $Z = 0.015$) $\mu \simeq 0.612$. For electrons, we have one particle per H nucleus, one-half per He nucleus and essentially all the particles as before for heavy elements (many electrons per ion). Hence:

$$\mu_e \simeq \frac{1}{[X + 0.5Y + 0.5(1 - X - Y)]} = \frac{2}{1 + X} \tag{16$'$}$$

Cloud fragmentation

In the cold ISM the Jeans' mass is of hundreds of $M_\odot$. Suppose at the beginning the cloud has $T = const$. Half the gravity ($3/10 GM^2/R$) is used (Virial theorem) to break solids and molecules (*non-thermal* degrees of freedom). In eq. (11), when ρ increases, M_J rapidly decreases, down to typical stellar masses. Hence every perturbation of ρ induces fragmentation and a cluster forms.

Do these fragments reach virial equilibrium? If so, they would be described by polytropes of exponent $\delta \geq 4/3$. For them:

$$T \propto \rho^{\delta-1} \Rightarrow M_J \propto \rho^{\frac{3\delta-4}{2}}$$

If $\delta = \gamma = 5/3$, then:

$$M \propto \rho^{1/2} \tag{17}$$

Denser fragments form more massive stars.

Deuterium burning

When a cloud collapses forming a protostar, the temperature at the centre easily reaches 1MK. In such conditions the small abundance of deuterium collected from the ISM (produeced e.g. in the Big Bang), undergoes proton captures:

$$^2D + p \rightarrow ^3 He + \gamma$$

A nuclear reaction is a sort of change of state: a process changing the absorption coefficient k_ν (see the heat tranfser part). Then k_ν increases and a convective instability affects the star. Convection brings more fuel to D burning and the star brightens, appearing in the HR diagram along the track characteristic of convective structures (Hayashi line).

Degeneracy

When a cloud contracts and finally reaches equilibrium, nuclear reactions can maintain the star stable for a time of the order of τ_{nucl}, which is of several billions of years for the Sun.
Onnset of nuclear energy generation not straightforward: requires Quantum Mechanical effects (tunnel effect). This needs a minimum temperature (around 10^7 K) hence **a minimum mass: about 0.08 $M_\odot$**. What is going on in lower mass clouds?
The temperature does not grow sufficiently, because the gravity is too low.
Density, instead, continues to increase as half the gravity is lost through radiation (virial theorem).
Stars become denser if of*lower* mass.
Particles will end up in being *in contact* with each other.

Degeneracy, continued

Let's suppose the particles are in contact, each occupying a volume $\sim l^3$ if l is their mean free path.

$$l \sim \left[\frac{1}{n}\right]^{1/3} = \left[\frac{m_p}{\rho}\right]^{1/3}.$$

An estimate for l can be taken from the De Broglie's wavelength λ_{dB} for a particle of velocity v, then v can be taken from the Maxwell distribution (very roughly!):

$$\lambda_{dB} = \frac{h}{m_p v}; \quad v \simeq \sqrt{\left[\frac{3kT}{m_p}\right]}$$

Simple algebra then gives you, for the cirtical density:

$$\rho_{deg} = const \cdot m^{5/2} T^{3/2} \tag{18}$$

Degeneracy, continued

Above the critical density matter does not behave as a perfect gas: particles are in contact. For T values typical of stars immediately after the MS (a few 10^7 K) and for particles of mass *m* equal to the electron mass, the critical density is about 10^6 g/cm^3.
The density is higher for higher mass particles: electrons becomne a *degenerate* gas for much lower densities than protons or neutrons!
A star encountering electron degeneracy before settling on the MS will evolve at very high central density, radiating its gravitational energy with no H-burning.
These objects are called *brown dwarfs*.
At their lower mass limit you find giant planets, like Jupiter.

Electron pressure and equation of state in white dwarfs

Electron degeneracy can be reached after H burning has occurred in normal stars (e.g. RGB, core He-flash, see lectures)

Degenereate cores of moderately massive stars are left at the end of the evolution as *white dwarfs*.

This is the fate of the Sun and of all stars less massive than 6-8 $M_\odot$!

In degenerate stars electrons exert an enormous pressure and ions are irrelevant in maintaining the stellar equilibrium.

IMPORTANT. A generic polytropic transformation for a Maxwell-Boltzmann's plasma is:

$$P = K(T, \mu)\rho^\delta \tag{19}$$

IN WHITE DWARFS T HAS DISAPPEARED. $P = K\rho^\delta$, with K = const., is an equation of state!