Hence:

The non-ideal gas case

As is shown, e.g., in Chandrasekhar (1939), for a non-ideal gas there is no longer a unique adiabatic exponent. In particular:

$$\chi_\rho = \frac{\partial \log P}{\partial \log \rho} = \Gamma 1 \qquad \frac{1}{\chi_T} = \frac{\partial \log T}{\partial \log P} = \frac{\Gamma 2 - 1}{\Gamma 2} \qquad (21)$$

In the bulk of a convective layer the polytropic equilibrium is very close to an adiabatic situation, so that there $\Gamma 2 \approx \Gamma 1 \approx \gamma$ **(in this case, as γ=5/3, ∇_{ad} is close to 0.4)**. If we can make this simplification, then radiative transport is established, AT LEAST, when:

$$\nabla_{rad} < \nabla_{ad} \approx \nabla_{conv} .$$

This condition is called **«the Schwarzschild's criterion»**

Otherwise one has, separately:

$$\mathbf{P = k_1 \rho^{\Gamma 1}}. \qquad \mathbf{P = k_2 T^{(\Gamma 2/(\Gamma 2-1))}}. \qquad (22)$$

In general, in other regions:

$$\Gamma 2 - \Gamma 1 = f(\beta(T)) \qquad (23)$$

Semiconvection.

The above holds in particular at the borders (not in the bulk mass) of convective regions. As shown by Langer (1983) the importance of radiation increases there, so that one has to define a more general gradient (Ledoux' gradient):

$$\nabla_L = \nabla_S + \frac{\beta}{4-3\beta}\nabla_\mu \qquad where \quad \nabla_\mu = \frac{\partial \log \mu}{\partial \log P} \qquad (24)$$

This is equivalent to say that the behavior of the plasma mimics a perfect gas only if the molecular weight has a vanishing derivative with respect to pressure ($\nabla_\mu = 0$).

In the zone not covered by the S. criterion, but by the **Ledoux' criterion** ($\nabla_{rad} < \nabla_L$), some form of «convection» occurs, but only partially, as the mean molecular weight varies. This boundary interval is called the «**semiconvective**» zone. There, energy is transported by radiation, but partial mixing changes slowly the composition. As in other similar slow phenomena in physics, we can use a diffusion equation (Fick's law) to approximately describe the evolution of the concentration ϕ of a certain component:

$$\frac{\partial \phi}{\partial t} = \nabla \cdot (D \nabla \phi) \qquad (25)$$

Where *D* is a coefficient of diffusion.

There exists an exact solution for this, which is the **error function** of argument $z = \frac{r}{\sqrt{4Dt}}$ (26)

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

and can then be approximated at first orders

A more general view. I

Although we cannot discuss the details for a question of time, it is obvious that relation (12) to (14) do not tell the whole story, as the equation of state also depends on the molecular weight, so that, together with:

$$\chi_T = \left(\frac{\partial \log P}{\partial \log T}\right)_{\rho,X_i} = \frac{T}{P}\left(\frac{\partial P}{\partial T}\right)_{\rho,X_i} \qquad \chi_\rho = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{T,X_i} = \frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_{T,X_i} \qquad (13)$$

we also have:

$$\chi_\mu = \left(\frac{\partial \log P}{\partial \log \mu}\right)_{T,X_i} = \frac{\mu}{P}\left(\frac{\partial P}{\partial \mu}\right)_{T,X_i} \qquad \nabla_\mu = \frac{\partial \log \mu}{\partial \log P} = \frac{1}{\chi_\mu} \qquad (27)$$

As shown e.g. in the paper by Salaris and Cassisi, 2017 (which I shall share as helpful material), by studying in this case the motion of a bubble like the one shown previously in Figure 1, one can derive equations for the differences ΔT, Δr, $\Delta\mu$, $\Delta\rho$ between the bubble's and the environment's materials. Their solutions imply that at least one of the following conditions

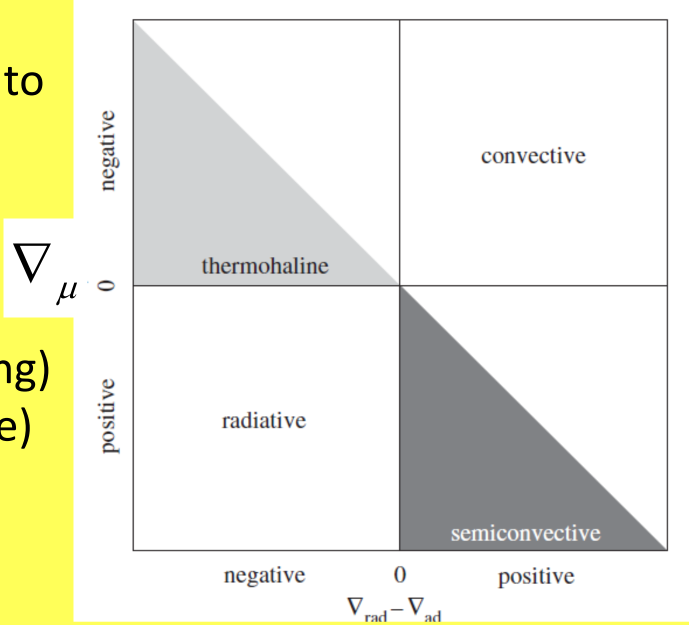
A more general view. II.

is satisfied: $\Delta\mu < 0$ (Rayleigh-Taylor instability)

$$\nabla_{rad} > \nabla_{ad} - \frac{\chi_\mu}{\chi_T}\nabla_\mu \equiv \nabla_L$$

$$\nabla_{rad} > \nabla_{ad}$$

If the first is true, then the molecular weight decreases toward the internal of the «bubble» and the situation is unstable. If the bubble climbs a distance Δr, and still has a density lower that the medium, it will continue to ascend until this is compensated.
If instead one has Δμ ≥ 0, then one of the two next conditions apply. They express the Schwarzschild's and Ledoux' criteria. A discussion of the solutions leads then us to consider the attached figure, where the various conditions are shown. While the lower left side is stable, in the others some bubble movement, hence mixing, will occur. Pure convection (full fast mixing) is the condition for the upper right (white) triangle, while the shaded zones refer to partial mixing cases.



Convective overshooting

In the discussion of the previous slides (as well as in the lecture by prof. El Eid) we defined as «convective» the zones where a bubble motion would be amplified. Its boundary corresponds therefore to the layer where this does not occu any longer.

However to have an amplification of a perturbation we need locally a force, hence an acceleration. Hence we set the boudary (Schwarzschild's criterion) where such a force would vanish.

Zero force does not however mean zero velocity. For examples, bubbles of the convective envelope in the Sun will reach the envelope base at zero acceleration, but with a residual velocity, forcing them to penetrate into the underlying radiative layers (the «tachocline»).

The velocity is in this case ofthen assumed not to go to zero abrupotly, but in some smooth form, e.g. with negative exponentials. This is e.g. the treatment done in the Italian code (name FRANEC, then FUNS) , whose stellar models are on-line at the site **FRUITY** (http://fruity.oa-teramo.inaf.it/).

The Mixing Length Theory

We often said that convective regions in their main part are «almost adiabatic». How is this treated in stellar models?

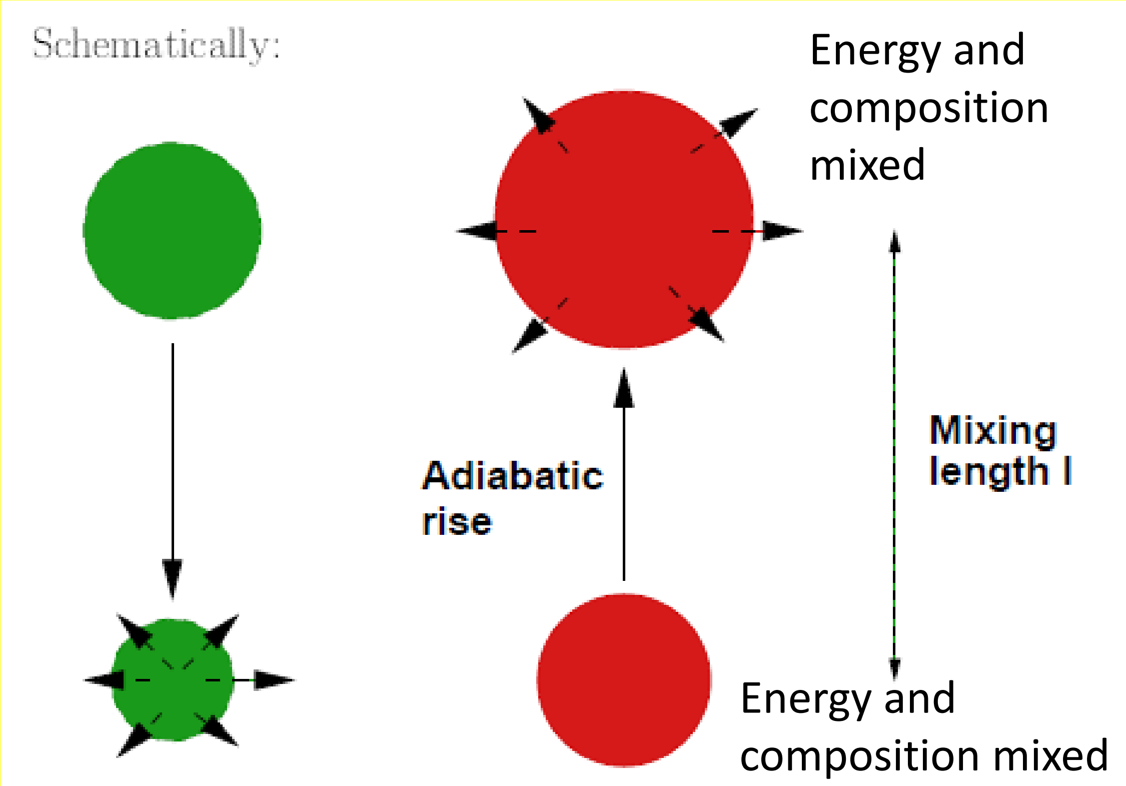
The simplest and most common approach is to follow the so-called Mixing Length Theory, early formulated by Prandtl for fluids (1915).

A clear presentation of the theory is availble at: [Prandtl's Mixing-Length Theory - an overview | ScienceDirect Topics].

Here we simply notice that it is based upon assuming that bubbles move adiabatically for a certain length *l* (the Mixing Length). After that, they are assumed to break releasing their energy and composition into the environment (fully adiabatic motions would be useless to transport energy). The Mixing Length is usually parameterized as a constant α (generally between 1 to 2) times the so-called *pressure scale height* λ_P

$$l = \alpha \lambda_P \qquad\qquad \lambda_P = \frac{P}{\left|\frac{dP}{dr}\right|}$$

Convection: the MLT. II.



A scheme for the Mixing Length Theory