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INFN - Section of Perugia

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Structure and equilibrium of normal stars

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ChETEC-SNAQs School on Nuclear Astrophysics Questions

Introduction

Trying to present stellar structure in 45 minutes does not make sense.
Therefore I shall not do that.
Rather, I'll briefly introduce some basic concepts to guide you to afford the subject on your own, later.
But later you will also forget what I said.
Hence, I'll provide some integration material where you can find the same stuff a little more expanded and explained.
This integration material is left to the organizers, so that it can be uploaded on the site of the school and made available,
When you read it again, for whatever doubt you may have please contact me freely at mauriziombusso@gmail.com.
I'll try to clarify.

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NOTES ADDED

Notes 1 - Similar to the present ones but more extended. Also some evolution & degeneracy

Notes 2 - More in a book-like form, from Onno Pols

Notes 3 - Something on stellar models and their integration from R. Nelson.

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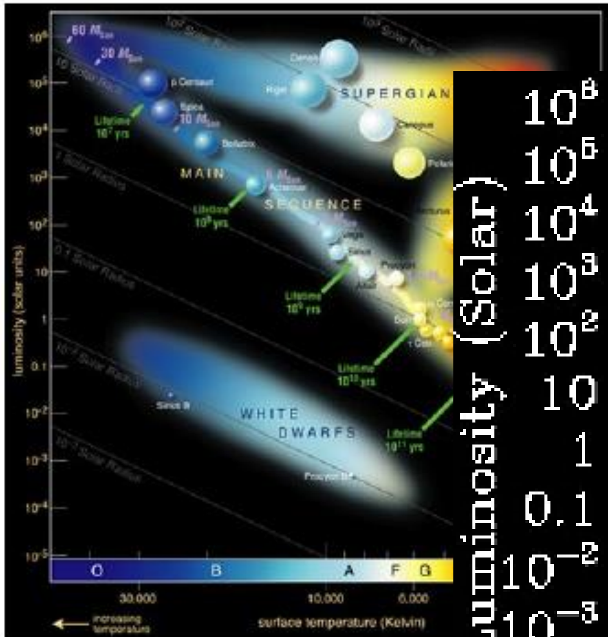
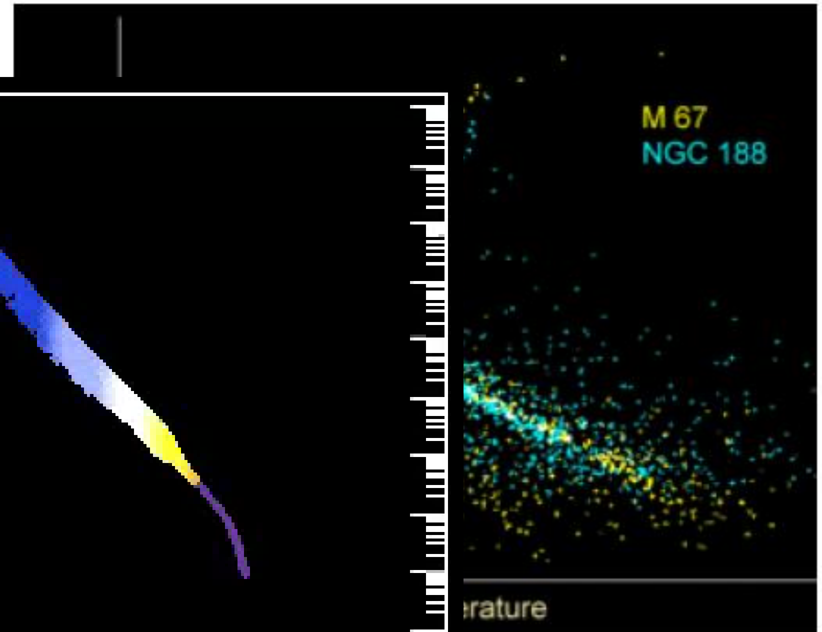
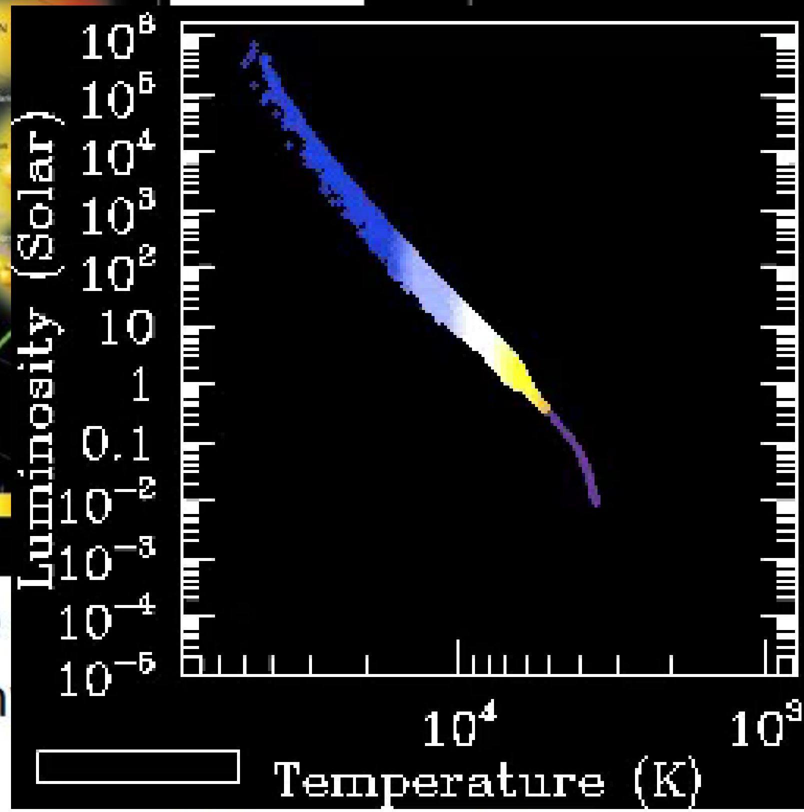
The Hertzsprung-Russell Diagram







Stellar structure … opens in the stars

Why stable? Wh… diagrams (i.e. 2 parameters)?

Clusters: best tool to calibrate models.

Interirors: fusion energy production & synthesis of elements.

Stars are the cauldrons where modern Alchemistry has its site!.

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CONTENT

The *free fall* time scale

Consider a mass element dM in a cloud of radius r_0 and a velocity $v = dr/dt$. Its *Potential Energy* is:

$$dU = -\frac{GM(r_0)dM}{r_0}$$

Energy is preserved under a *conservative* force like gravity. Hence, if U_i, U_f and K_i, K_f are the potential and kinetic energies at the beginning and at the end, respectively:

$$U_i + K_i = U_f + K_f$$

Consider "the end" to be the case in wich the cloud, after collapsing, comes to a full stop at a radius r. Then:

$$-\frac{GM(r_0)dM}{r_0} + \frac{1}{2}dM\left(\frac{dr}{dt}\right)^2 = -\frac{GM(r_0)dM}{r}$$

Free fall of the solar cloud

Now let's look for the *free fall* timescale, i.e. for the time taken to reach a radius close to zero ($r_f \simeq 0$), through a *homologous* unhampered fall. You easily get:

$$\left(\frac{dr}{dt}\right) = \left[-2GM(r_0)\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{1/2}$$

or:

$$\tau_{ff} = \int_{r_0}^{0} dt = \int_{r_0}^{0} \left[2GM(r_0)\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr$$

which, if ρ_0 is the initial density, provides the result:

$$\tau_{ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{1/2} \qquad (1)$$

You can verify that, for a typical initial density of 3×10^3 particles/cm^3, this gives a time scale of about 0.5 Myr.

The Kelvin Helmoltz Time Scale

How long would the Sun remain active, if its main energy source were gravitation?. We can say that:

$$\Omega = -G \int_0^R \frac{MdM}{R(M)} = -\frac{3}{5}\frac{GM^2}{R}$$

which is easily obtained by substituting for $R(M)$ the relation you get if the Sun is spherical:

$$dM \simeq 4\pi\rho R^2 dR$$

From the virial theorem at equilibrium ($U + 2\Omega = 0$; see integrations), up to half that energy can be radiated, producing the solar luminosity L (4×10^{33} erg/sec), hence:

$$\Delta t = \frac{3}{10}\frac{GM^2}{RL} \simeq 20 Myr \tag{2}$$

Energy and lifetime through nuclear fusion

As you know, Einstein's equation for the equivalece of mass and energy says that:

$$E = mc^2$$

In the twenties of the XXth century, the mass of the He nucleus was measured carefully, being 4,002602 a.m.u. Sir Eddington then noticed that the fusion of four protons would produce 4×1.007=4.028 a.m.u., i.e. a He mass, plus an excess of about 0.007 times the He mass itself. This implies that, if the Sun burns about 10% of its mass, this guarantees a duration of the process of:

$$\Delta t = \frac{\Delta M c^2}{L} \simeq \frac{0.1 M_{\odot} * 0.007 * M_{He} * c^2}{4 \times 10^{33}} \simeq 10 Gyr \qquad (3)$$

Since then, we know the Sun is powered by H burning into He.

Polytropes: a simple introduction

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The thermodynamic concept of "polytropic transformations" (Chandrasekhar 1958) is a crucial one in physics. For a stirred perfect gas, this concept is used when, in the stirring, $dQ = cdT$, T being the temperature in K and c a constant *(specific heat)*. If the energy is preserved, we know that the equation of the adiabatic transformations holds:

$$P \propto \rho^{\gamma} \tag{4}$$

where ρ is the density, P is the pressure and γ is the adiabatic exponent ($\equiv 5/3$ for a perfect gas with 3 degrees of freedom). More generally, we can show that in polytropes a relation:

$$P \propto \rho^{\delta} \tag{5}$$

remains valid, even when $\delta \neq \gamma$. For $c = 0, \delta = \gamma$ the process is adiabatic, for $\delta = 1$ it's isothermal.

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Adiabatic exponent(s)

The polytropic exponent δ, or the adiabatic one γ, are defined for perfect gas transformations like:

$$P \propto \rho^{\gamma}; \quad T \propto \rho^{\gamma-1} etc... \tag{6}$$

Differentiating them one has:

$$\frac{dT}{T} = (\gamma - 1)\frac{d\rho}{\rho}; \quad \frac{dP}{P} = \gamma\frac{d\rho}{\rho}; \quad \frac{dP}{P} = \frac{\gamma}{\gamma - 1}\frac{dT}{T} \tag{6'}$$

This holds for a perfect gas.
What happens if the pressure is due also to radiation?
For an adiabatic transformation one can still derive similar relations, but the exponents are no longer the same as γ for a pure gas. Relations like (6') have different exponents ($\Gamma_1, \Gamma_2, \Gamma_3, ...$; Chandrasekhar, 1958; see page 19).

Pressure equilibrium

For a nebula in a state of motion under its gravity, one has:

$$\rho \frac{\partial^2 r}{\partial t^2} + \frac{\partial P}{\partial r} = -G\frac{M(r)\rho}{r^2}$$

At equilibrium,

$$\frac{dP}{dr} = -\rho g = -G\rho\frac{M(r)}{r^2} \tag{7}$$

For a spherical star,

$$dM = 4\pi\rho r^2 dr, \quad r = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$$

This can be written as a further equation (conservation of spherical mass in *Eulerian* coordinates, using the radius r as a variable):

$$\frac{dM}{dr} = 4\pi\rho r^2 \tag{8}$$

It can also be used together with pressure equilibrium to get:

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \tag{9}$$

This is an inelegant, but quick, way to express pressure equilibrium in *Lagrangian* coordinates.
Stellar models use this, with *mass* as a variable: mass varies less radically than radius, allowing easier integration with better resolution.
In this simple exposition, however, we maintain the Eulerian form.
We need now more equations linking the variables.
One of them is the *Equation of State*, where thermodynamic variables are linked to *microscopic* properties, like the mean mass of the particles,called *mean molecular weight*.

The mean molecular weight

In a fully ionized gas, let the mass fractions of H, He and heavier nuclei be X, Y and Z. Let's count the number of free particles. For hydrogen: $n_H = X/m_H$; $n_e = X/m_H$. For helium: $n_{He} = Y/4m_H$; $n_e = Y/2m_H$. For each heavy nucleus: $n_{Z_i} = X_i/A_i m_H$; $n_e = X_i Z_i/A_i m_H$. Hence:

$$n_{V,i} = \frac{\rho X_i}{A_i m_H} \qquad n_{m,i} = \frac{n_{V,i}}{\rho} = \frac{X_i}{A_i m_H}$$

Then we notice that, for heavy elements, $(Z_i/A_i)_{aver} \simeq 1/2$ and $\Sigma X_i/A_i << 1$. Then:

$$n = \left(2X + \frac{3Y}{4} + \frac{Z}{2}\right) \times \frac{\rho}{m_H}$$

Now, as:

$$n = \frac{\rho}{\mu m_H} \rightarrow \mu = \frac{1}{2X + 3/4Y + Z/2} \qquad (10)$$

The Equation of State

In a perfect gas we know that: $PV = nRT$ (with $R = 8.3144$ $Kg \cdot m^2 \cdot sec^{-2} \cdot K^{-1} \cdot mol^{-1}$). This can be rewritten as:

$$P = n(k_B N_A)T/V$$

where $k_B = 1.3810^{-23}\ J/K$ is the Boltzmann's constant and N_A is the Avogadro's number ($6.023\ 10^{23}$). By multiplying up and down by the mass M one gets:

$$P = nk_B \frac{N_A}{M}\frac{M}{V}T = n\frac{k_B}{\mu m_H}\rho T \qquad (11)$$

where the mean mass of each particle, M/N_A is set to μm_H. Here μ is the mean molecular weight just defined, and m_H is the value of the a.m.u. ($1.67\ 10^{-27}$ Kg). This is the gas pressure. Then:

$$P = P_g + P_r = \frac{k_B \rho T}{\mu m_H} + \frac{1}{3}aT^4 \qquad (12)$$

Radiative Transfer of Energy

When energy is transported by the interaction of photons with matter (photons being absorbed and re-emitted by each stellar layer, with a very short mean free path) the fraction of the flux dF_ν across a layer of depth dr will be proportional to the flux itself F_ν; the coefficient $\bar{k}_\nu$ describes the interaction between radiation and matter per unit volume. Using the coefficient per unit *mass* k_ν ($\bar{k}_\nu = k_\nu \rho$):

$$dF_\nu = -F_\nu k_\nu \rho dr$$

Hence:

$$\frac{dF_\nu}{dr} = -k_\nu \rho F_\nu \tag{13}$$

And since $dP_r = dF_r/c$,

$$\frac{dP_r}{dr} = -\frac{k_\nu \rho L(r)}{4\pi c r^2} \tag{14}$$

Radiation Transfer (cont.)

If we express the pressure gradient with respect to the temperature gradient, we obviously have:

$$P = \frac{1}{3}aT^4 \rightarrow \frac{dP_r}{dr} = \frac{4a}{3}T^3\frac{dT}{dr}$$

and comparing this with the previuous relation, one gets:

$$\frac{dT}{dr} = -\frac{3k\rho L}{16\pi acT^3r^2} \qquad (15)$$

It is useful to derive also a logarithmic ratio with respect to pressure (as we shall see shortly at page 18). As:

$$\frac{dP}{dr} = -G\frac{M\rho}{r^2}$$

this turns out to be (see the attached notes):

$$\left(\frac{dlogT}{dlogP}\right)_{rad} = \frac{3kLP}{16\pi acGMT^4} \qquad (15')$$

Convective transfer of Energy

When the coefficient k_ν becomes too large the heat is more easily transported by convective motions.
Assuming a perfect gas, we know relations like ('6) are valid. In particular:

$$\frac{dP}{P} = \frac{\delta}{\delta - 1}\frac{dT}{T}; \quad \frac{dlogT}{dlogP} = \frac{\delta - 1}{\delta} \tag{16}$$

When the gas is almost adiabatic, $\delta \simeq \gamma$, and;

$$\nabla = \left(\frac{dlogT}{dogP}\right)_{ad} = \frac{\gamma - 1}{\gamma} \tag{16'}$$

Adiabaticity says this is the minimum gradient (zero transport). Reality will be different.
In slide (18) we'll see how to decide which mechanism transports energy.

The Equations of a Stellar Model

Summing up, we have equations for density equilibrium, mass conservation, temperature gradient:

$$\frac{dP}{dr} = -\frac{4\pi G\rho^2 r}{3} \tag{17}$$

$$\frac{dM}{dr} = 4\pi\rho r^2 \tag{18}$$

$$\frac{dLogT}{dlogP} = \nabla_{conv}; \frac{dLogT}{dlogP} = \nabla_{rad} \tag{19}$$

Plus, we have energy generation, let's indicate it as a function $\epsilon(r)$ of the variables:

$$\frac{dL}{dr} = 4\pi r^2 \rho\epsilon \tag{20}$$

Then we have auxiliary, non-differential, equations:

$$\epsilon = \epsilon(r, T, P, \rho, \mu) \tag{21}$$

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Equations & Boundary Conditions

$$\mu = \mu(r, T, P, \rho, \mu) \tag{22}$$

$$P = \frac{k_B}{\mu m_H} \rho T \tag{23}$$

$$k_\nu = k_\nu(r, T, P, \rho, \mu) \tag{24}$$

From (17) to (24) we have 8 equations.
The boundary conditions must be known: 4 for the normal equations; twice that for the differential equations.
Instead, for two critical functions (pressure and temperature) we know the surface conditions, but not the central ones. Two parameters are “free ”.
And indeed, while a star evolves, its representative point can move in the 2D H-R diagram, describing evolutionary tracks!
Stellar evolution has two main degrees of freedom.

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Criteria for Convection

A way to establish what is the energy transport mechanism is to compare ∇ to its equivalent in radiative conditions, ∇_{rad} (15'): the lower of the two is the one established, for the second principle of thermodynamics.
This criterion (the Schwarzschild's criterion) is simple but it's not very good.
We see this if we recall the expression (12) of pressure with radiation.
This is the sum of two terms. Let's assume for the first one the adiabatic law (3 deg. of freedom): $P = K\rho^{\gamma} = K\rho^{5/3}$.
Its equivalent for radiation derives from the relation $P = 1/3U$ (U = energy density: see attached notes).
It can be shown in this way that $P_r = K_1\rho^{4/3}$.
Hence for the pressure we sum two polytropes with different exponents. What's the outcome??

Ledoux's criterion. Semiconvection

When radiation is important μ is variable and we know that there are different polytropic exponents (slide 8). In particular:

$$\frac{dlnP}{dln\rho} = \Gamma_1; \quad \frac{dlnP}{dlnT} = \frac{\Gamma_2}{\Gamma_2 - 1}; \quad \frac{dlnT}{dln\rho} = \Gamma_3 - 1 \qquad (3)$$

We can also define a gradient versus the molecular weight:

$$\nabla_\mu = \frac{dlnT}{dln\mu}$$

At the borders of convective zones one finds layers where bubbles can move adiabatically, but heat is transported by radiation, when:

$$\nabla_{rad} < \nabla_{ad} + k\nabla_\mu \qquad (17)$$

These regions are called *semiconvective* and (17) is called "Ledoux's criterion".

THE END. THANKS FOR ATTENTION!!