Numerical Solution of Nonlinear Second-Order Boundary Value Problems Using Vashakmadze's Method with Application to Elastic Beam Bending

GGBC 2025 conference: QUALI-start-up

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About Me

- Master student at **Ivane Javakhishvili Tbilisi State University**, final year.
- Participant in two international conferences:
  - *International Enlarged Sessions of the Seminar of Ilia Vekua Institute of Applied Mathematics of Ivane Javakhishvili Tbilisi State University* (Tbilisi, 2025).
  - *International Conference on Probability Theory and Statistics* Dedicated to the 80th Anniversary of Professor Estate Khmaladze (Tbilisi, 2025).
- Article publication:
  - **"Regression Empirical Process and Distribution-Free Transformation"**, in *Applied Mathematics, Informatics and Mechanics*, Vol. 30, No. 1, 2025.

Bending Equation and Methods of Solution

$$u''(x) = \frac{M(x)}{EI}\left[1 + \left(u'(x)\right)^2\right]^{3/2}$$

- $u(x)$ — deflected shape of the beam's neutral axis;
- $M(x)$ — internal bending moment;
- E — Young's modulus of the material;
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Euler–Bernoulli theory:

$$\left[1 + \left(u'(x)\right)^2\right]^{3/2} \approx 1,$$

which reduces the equation to the linear form

$$u''(x) \approx \frac{M(x)}{EI}.$$

Second-Order Nonlinear Differential Equation

$$u''(x) = f(x, u(x), u'(x)), \quad x \in [0, 1],$$
$$k_1 u(0) + k_2 u'(0) = \alpha, \quad k_3 u(1) + k_4 u'(1) = \beta,$$

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$$\exists\, L, L' > 0 : \quad \left| f(x, u_1, v_1) - f(x, u_2, v_2) \right| \le L \left| u_1 - u_2 \right| + L' \left| v_1 - v_2 \right|.$$

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- $L + L' < 4$;
- $\dfrac{L}{8} + \dfrac{L'}{2} < 1$.

Main Formulas

$$u(x) = \frac{x_p - x}{x_p - x_1} u(x_1) + \frac{x - x_1}{x_p - x_1} u(x_p) + \frac{1}{x_p - x_1} \times$$

$$\times \left[(x_p - x_1) \int_{x_1}^{x} \int_{x_1}^{t} f(s, u(s), u'(s))\, ds\, dt - (x - x_1) \int_{x_1}^{x_p} \int_{x_1}^{t} f(s, u(s), u'(s))\, ds\, dt \right].$$

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$$f(s, u(s), u'(s)) \approx \sum_{j=2}^{p-1} f\big(x_j, u(x_j), u'(x_j)\big)\, \ell_j(s),$$

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$$b_{ij} := \frac{1}{x_p - x_1} \left[(x_p - x_1) \int_{x_1}^{x_i} \int_{x_1}^{x} \ell_j(t)\, dt\, dx - (x_i - x_1) \int_{x_1}^{x_p} \int_{x_1}^{x} \ell_j(t)\, dt\, dx \right].$$

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$$b_{p+1-i,\, p+1-j} = b_{ij}, \quad i = 2, \ldots, \frac{p+1}{2}, \quad j = 2, \ldots, p-1,$$

Main Formulas

$$u'(x) = \frac{1}{x_p - x_1}\left(u(x_p) - u(x_1)\right) - \frac{1}{x_p - x_1}\int_0^{x_p}\int_x^t f(s, u(s), u'(s))\, ds\, dt.$$

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replace $f(s, u(s), u'(s))$ with Lagrange polynomial.

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$$\boxed{u'(x_i) \approx \frac{u(x_p) - u(x_1)}{x_p - x_1} - \frac{1}{x_p - x_1}\sum_{j=2}^{p-1} c_{ij}\, f(x_j, u(x_j), u'(x_j))}.$$

Classification of Grid Points

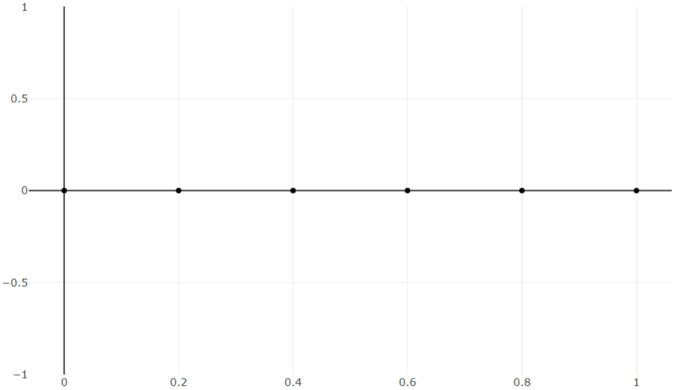

Figure: Sub-intervals

Classification of Grid Points

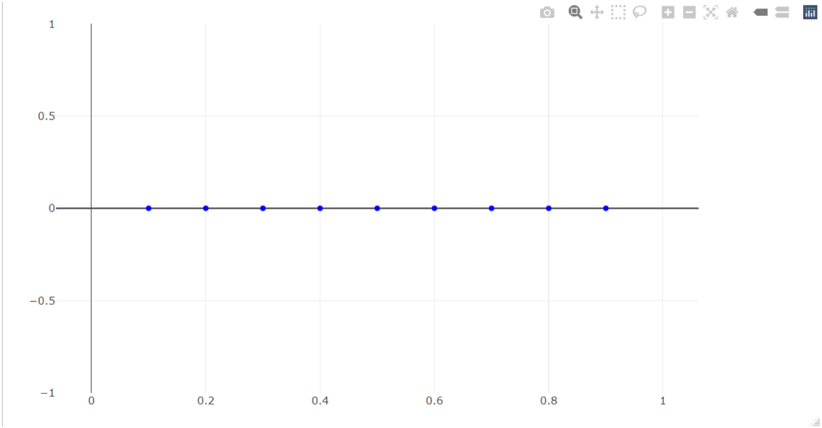

Figure: Central points: x_{ts+1}, $t = 1, 2, \ldots, 2k - 1$.

Classification of Grid Points

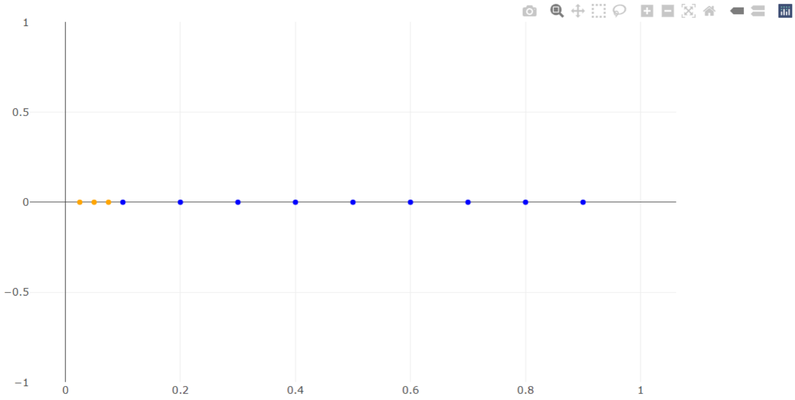

Figure: Left boundary points: $x_j, \qquad j = 2, 3, \ldots, s.$

Classification of Grid Points

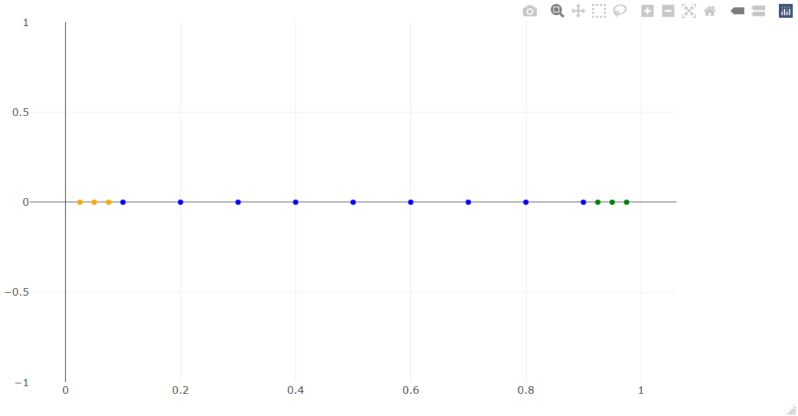

Figure: Right boundary points: $x_j, \qquad j = (2k-1)s+2, \ \ldots, \ 2ks.$

Classification of Grid Points

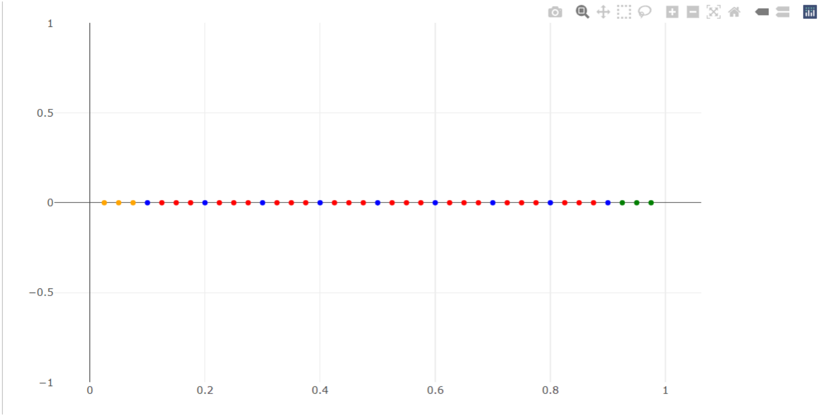

Figure: Middle points.

Equations for Central Nodes

$$u_{ts+1} = \frac{1}{2}u_{(t-1)s+1} + \frac{1}{2}u_{(t+1)s+1} + A_t, \quad t = 2, \ldots, 2k-2$$

$$A_t = \sum_{j=2}^{2s} b_{s+1,j} \cdot u''_{(t-1)s+j}.$$

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$$u_{s+1} = \frac{-1}{2(kk_2 - k_1)} \alpha + \frac{2kk_2 - k_1}{2(kk_2 - k_1)} \cdot u_{2s+1} + A_1$$

$$A_1 = \sum_{j=2}^{2s} \left(b_{s+1,j} - \frac{kk_2}{2(kk_2 - k_1)} c_{1,j} \right) \cdot u''_j.$$

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$$u_{(2k-1)s+1} = \frac{1}{2(kk_4 + k_3)}\beta + \frac{2kk_4 + k_3}{2(kk_4 + k_3)} \cdot u_{(2k-2)s+1} + A_{2k-1}$$

$$A_{2k-1} = \sum_{j=2}^{2s} \left(b_{s+1,j} + \frac{kk_4}{2(kk_4 + k_3)} c_{2s+1,j} \right) \cdot u''_{(2k-2)s+j}.$$

Calculation of A_t

$$u''(x_j)^{[0]} = f(x_j, 0, 0).$$

$$A_1^{[0]} = \sum_{j=2}^{2s} \left(b_{s+1,j} - \frac{kk_2}{2(kk_2 - k_1)} c_{1,j} \right) \cdot u_j''^{[0]};$$

$$A_t^{[0]} = \sum_{j=2}^{2s} b_{s+1,j} \cdot u''^{[0]}_{(t-1)s+j}, \quad t = 2, \ldots, 2k-2;$$

$$A_{2k-1}^{[0]} = \sum_{j=2}^{2s} \left(b_{s+1,j} + \frac{kk_4}{2(kk_4 + k_3)} c_{2s+1,j} \right) \cdot u''^{[0]}_{(2k-2)s+j}.$$

Node Located in the Center of Domain x_{ks+1}

$$A := \frac{2kk_2 - k_1}{2(kk_2 - k_1)}, \qquad B := \frac{2kk_4 + k_3}{2(kk_4 + k_3)}.$$

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$$\begin{aligned}
N_1 &= \big((k-1) - B(k-2)\big) \cdot 1,\\
N_2 &= \big((k-1) - B(k-2)\big) \cdot 2,\\
N_i &= (k-2)\cdot 2(i-2)AB + (1-k)\cdot 2(i-2)A\\
&\quad + (2-k)\cdot 2(i-1)B + (k-1)\cdot 2(i-1), \quad \text{for } 3 \le i \le k,\\
N_i &= (k-2)\cdot 2(2k-i-2)AB + (1-k)\cdot 2(2k-i-2)B\\
&\quad + (2-k)\cdot 2(2k-i-1)A + (k-1)\cdot 2(2k-i-1), \quad \text{for } k < i \le 2k-3,\\
N_i &= \big((k-1) - A(k-2)\big)\cdot(2k-i), \quad \text{for } i = 2k-2, 2k-1.
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\end{aligned}$$

$$u^{[n]}_{ks+1} = \frac{N_1}{D} \cdot \frac{-1}{2(kk_2 - k_1)}\alpha + \frac{N_{2k-1}}{D} \cdot \frac{1}{2(kk_4 + k_3)}\beta + \sum_{i=1}^{2k-1} \frac{N_i}{D} A_i^{[n-1]}$$

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$$\begin{aligned} \widetilde{N}_1^B(t) &= -1, \\ \widetilde{N}_2^B(t) &= -2, \\ \widetilde{N}_i^B(t) &= 2(i-2)B - 2(i-1), \quad \text{for } i = 3, \dots, t. \end{aligned}$$

$$u_{(2k-t)s+1}^{[n]} = \frac{1}{2(kk_4+k_3)} \cdot \frac{\widetilde{N}_1^B(t)}{\widetilde{D}^B(t)} \cdot \beta + \frac{\widetilde{N}_t^B(t)}{2\widetilde{D}^B(t)} \cdot u_{(2k-t-1)s+1}^{[n]} + \sum_{i=1}^{t} \frac{\widetilde{N}_i^B(t)}{\widetilde{D}^B(t)} A_{2k-i}^{[n-1]}, \quad t = k-1, \dots, 2.$$

Recursive Formulas for Central Nodes

$2 \le t \le k-1, \quad u_{(2k-t)s+1}$

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$$u^{[n]}_{(2k-t)s+1} = \frac{1}{2(kk_4+k_3)} \cdot \frac{\widetilde{N}_1^B(t)}{\widetilde{D}^B(t)} \cdot \beta + \frac{\widetilde{N}_t^B(t)}{2\widetilde{D}^B(t)} \cdot u^{[n]}_{(2k-t-1)s+1} + \sum_{i=1}^{t} \frac{\widetilde{N}_i^B(t)}{\widetilde{D}^B(t)} A^{[n-1]}_{2k-i}, \quad t = k-1, \dots, 2.$$

$$u^{[n]}_{(2k-1)s+1} = \frac{1}{2(kk_4+k_3)}\beta + B \cdot u^{[n]}_{(2k-2)s+1} + A^{[n-1]}_{2k-1}.$$

Formulas for Non-Central Nodes

$i = 2, \ldots, s$

$$u_i^{[n]} = \frac{k(x_{2s+1} - x_i)}{k_1 - kk_2} \cdot \alpha + \left(k(x_i - x_1) - \frac{(x_{2s+1} - x_i)k^2 k_2}{k_1 - kk_2} \right) \cdot u_{2s+1}^{[n]} + \sum_{j=2}^{2s} \left(b_{i,j} + \frac{(x_{2s+1} - x_i)k^2 k_2}{k_1 - kk_2} \cdot c_{1,j} \right) \cdot u_j''^{[n-1]};$$

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$$u_{(2k-2)s+i}^{[n]} = \frac{(x_{(2k-2)s+i} - x_{(2k-2)s+1})k}{k_3 + kk_4} \cdot \beta + \left((x_{2ks+1} - x_{(2k-2)s+i})k + \frac{(x_{(2k-2)s+i} - x_{(2k-2)s+1})k^2 k_4}{k_3 + kk_4} \right) \cdot u_{(2k-2)s+1}^{[n]} + \sum_{j=2}^{2s} \left(b_{i,j} + \frac{(x_{(2k-2)s+i} - x_{(2k-2)s+1})k^2 k_4}{k_3 + kk_4} \cdot c_{2s+1,j} \right) \cdot u''^{[n-1]}_{(2k-2)s+j};$$

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Derivative Approximation

$t = 2, \dots, 2k-2, \quad i = 1, \dots, 2s+1$

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First Example

$$u''(x) = \frac{M(x)}{EI}\left(1 + \left(u'(x)\right)^2\right)^{3/2}, \quad x \in [0,1],$$

$$M(x) = \frac{EI}{\cosh^2(x)}.$$

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The total execution time 1.77 seconds.

Second Example

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After 22 iteration:

$$\max_i |u_i - u(x_i)| = 1.53 \times 10^{-8},$$

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$$\max_i |u_i' - u'(x_i)| = 5.14 \times 10^{-8}.$$

The total execution time 1.81 seconds.

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