Strong-field Quantum-Electrodynamics: from amplitudes to physical effects

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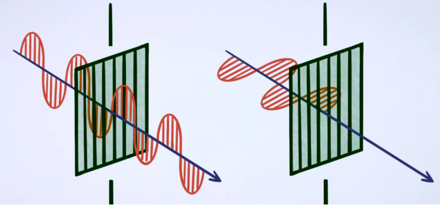
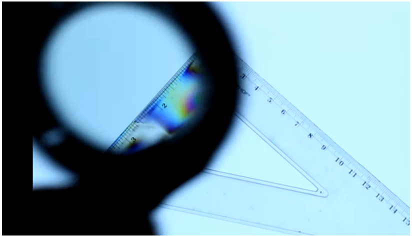
Polish-German WE-Heraeus Seminar & Max Born
Symposium-Many-particle systems under extreme conditions
Görlitz, 3-6 December 2023

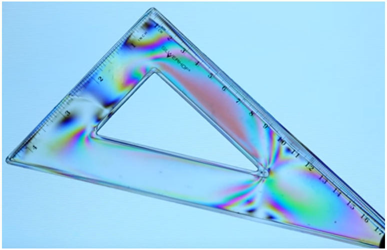
OUTLINE

- Introduction
- Euler-Heisenberg Lagrangian and the four-photon amplitude
- Vacuum birefringence
- Four-photon amplitude beyond Euler-Heisenberg
- Conclusion

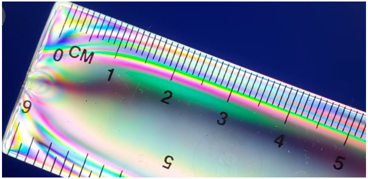
SIMPLE EXPERIMENT

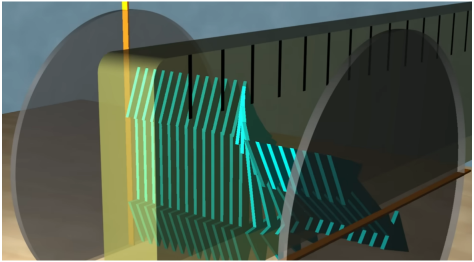






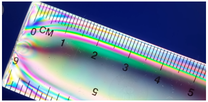
WHAT DO WE OBSERVE?

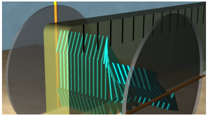




👉The ruler rotates the polarization of light wave!

VACUUM BIREFRINGENCE

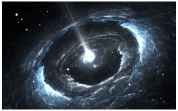




- Ruler ⇒ Vacuum
- Stress pattern ⇒ Pump Laser
- Rotated wave ⇒ Probe Laser

Why is it important?

- Testing Strong Field Quantum Electrodynamics.
- Manifestation of the quantum nature of the vacuum.
- Understanding of the behavior of light and matter in extreme conditions: Astrophysics.
- Testing standard model of particle physics (axions?).
- The prediction dates back to the 1930s, no direct observation yet.



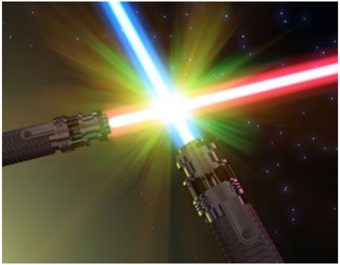
CLASSICAL ELECTRODYNAMICS

Maxwell Equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \quad \nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

Superposition principle:



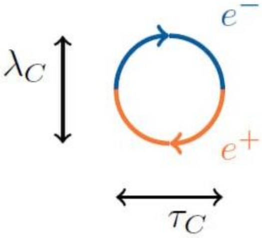
QUANTUM VACUUM

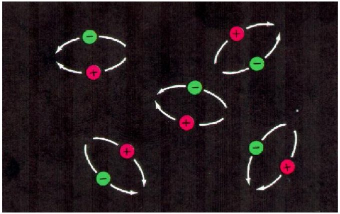
The quantum vacuum is complex, fluctuates, and can be polarized.

- Heisenberg: uncertainty principle $\Delta E \Delta t \geq \hbar/2$
- Dirac: every particle has anti-particle
- Einstein: $E = mc^2$ virtual particle-antiparticle pairs

Vacuum is filled with virtual electron-positron pairs.

- short life time: $\tau_C = \hbar/mc^2 = 10^{-21}s$
- small extent: $\lambda_C = \hbar/mc = 10^{-11}cm$



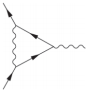


QUNATUM ELECTRODYNAMICS (THE JEWEL OF PHYSICS)

Nobel Prize 1965

Feynman diagrams

Quantum Electrodynamics in vacuum

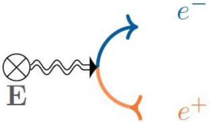


- Successful: theory/experiment
  (magnetic moment of electron: 10^{-13}, Lamb shift)
- Predictions based on $\Rightarrow$ perturbation theory (Feynman diagrams)

Quantum Electrodynamics in background fields

Electromagnetic backgrounds greatly modify quantum vacuum.

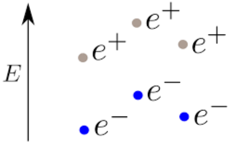
- Electron-positron pair production



$E_{\text{crit}} = m^2c^3/(e\hbar) = 1.3 \times 10^{16} V/cm$, $B_{\text{crit}} = 4.4 \times 10^9 T$

$I_{\text{crit}} = 4.6 \times 10^{29} W/cm^2$, $I_{\text{exp}} \sim 10^{23} W/cm^2$

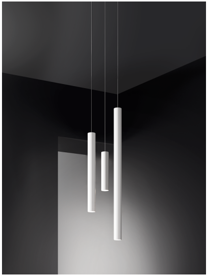
- Vacuum polarization



$I_{\text{exp}} \sim 10^{23} W/cm^2$

FIELD STRENGTHS: COMPARISON

Electric Field	V/m	W/cm^2
Lightning Tube	10	10^{-5}
Lightning strike	3×10^5	10^4
Strong lasers	$10^{15} - 10^{16}$	10^{24}
Schwinger field	10^{18}	10^{29}



EULER-HEISENBERG LAGRANGIAN

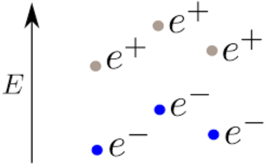
Constant Electromagnetic field: W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936)

$$\mathcal{L}_{\rm EH} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left\{\frac{(eaT)(ebT)}{\tan(eaT)\tanh(ebT)} - \frac{2}{3}(eT)^2\mathcal{F} - 1\right\}$$

$$a = \left(\sqrt{\mathcal{F}^2+\mathcal{G}^2} - \mathcal{F}\right)^{1/2}, \qquad b = \left(\sqrt{\mathcal{F}^2+\mathcal{G}^2} + \mathcal{F}\right)^{1/2}$$

with the two invariants

$$-2\mathcal{F} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \vec{E}^2 - \vec{B}^2, \qquad -\mathcal{G} = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = \vec{E}\cdot\vec{B}$$

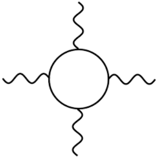


LOW-FIELD LIMIT

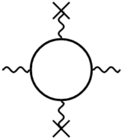
Photon-photon scattering:

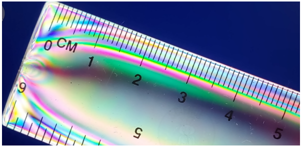
$$\mathcal{L}_{\mathrm{EH}} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \xi\left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2\right] + \cdots$$

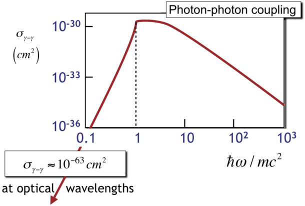
$$\xi = \frac{e^4}{360\pi^2 m^4} = \frac{\alpha}{90\pi E_{\mathrm{crit}}^2} \quad , \quad \hbar = c = 1$$



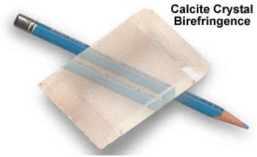
Vacuum birefringence:

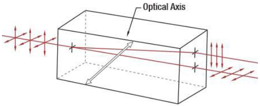


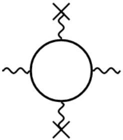




Adapted from: Tommasini et al.







Strong external fields $\Rightarrow$ Vacuum becomes nonlinear!

$$\begin{cases} n_{\parallel} = 1 + 14\frac{e^2}{45\pi^2}\frac{I}{I_{\text{crit}}} \\ n_{\perp} = 1 + 8\frac{e^2}{45\pi^2}\frac{I}{I_{\text{crit}}} \end{cases} \quad \Rightarrow \Delta = 2\pi\frac{L}{\lambda}(n_{\parallel} - n_{\perp})$$

- L: interaction length
- λ: the wavelength

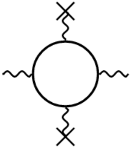
What do we need?

- Pump laser
- Probe laser

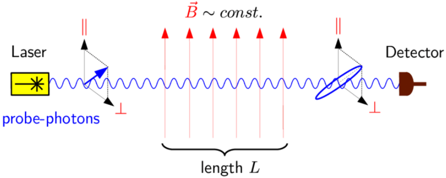
T. Heinzl, B. Liesfeld, K.-U. Amthor, H. Schwoerer, R. Sauerbrey, A. Wipf, Opt.Commun. **267** (2006) 318-321

PUMP & PROBE FIELDS

pump field (polarizes vacuum)		
magnetic field	laser focus	nuclear Coulomb field
$\mathcal{O}(10^{-9} B_{\text{crit}})$	$\mathcal{O}(10^{-4} E_{\text{crit}})$	$\mathcal{O}(E_{\text{crit}})$
$\delta n = \mathcal{O}(10^{-22})$	$\delta n = \mathcal{O}(10^{-11})$	$\delta n = \mathcal{O}(10^{-2})$
field strength $\rightarrow$		$\leftarrow$ interaction volume
probe field (detects vacuum polarization)		
optical laser	X-ray	γ-ray
$\mathcal{O}(\text{eV})$	$\mathcal{O}(\text{keV})$	$\mathcal{O}(\text{MeV})$
$N = \mathcal{O}(10^{20})$	$N = \mathcal{O}(10^{11})$	$N = \mathcal{O}(1)$
wavenumber $\rightarrow$		$\leftarrow$ photon number
PVLAS, BMV, ...	HIBEF	Delbrück (ATLAS)

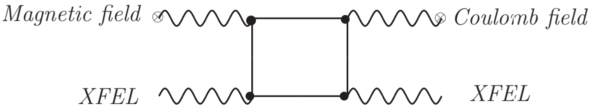


N. Ahmadiniaz, T.E. Cowan, R. Sauerbrey, U. Schramm, H.-P. Schlenvoigt, R. Schützhold., Phys. Rev. D **101**, 116019 (2020)



COULOMB-ASSISTED VACUUM BIREFRINGENCE

N. Ahmadiniaz, M. Bussmann, T.E. Cowan, A. Debus, T. Kluge, R. Schützhold, Phys. Rev. D Lett. **104**, 011902 (2021)



$$\mathcal{L}_{\rm LE} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \xi\left[(\mathfrak{E}^2 - \mathfrak{B}^2)^2 + 7(\mathfrak{E}\cdot\mathfrak{B})^2\right]$$

$$\mathfrak{E} = \mathbf{E}_{\rm ext} + \mathbf{E} \quad , \quad \mathfrak{B} = \mathbf{B}_{\rm ext} + \mathbf{B} \quad , \quad \xi = \frac{2\alpha^2_{\rm QED}}{45 m^4}$$

The effective Lagrangian for the XFEL fields:

$$\mathcal{L}_{\rm XFEL} = \frac{1}{2}[\mathbf{E}\cdot(\mathbb{1} + \delta\epsilon)\cdot\mathbf{E} - \mathbf{B}\cdot(\mathbb{1} - \delta\mu)\cdot\mathbf{B}] + \mathbf{E}\cdot\delta\Psi\cdot\mathbf{B}$$

External fields $\mathbf{E}_{\rm ext}$ and $\mathbf{B}_{\rm ext}$ $\Rightarrow$ medium properties

$$\delta\Psi^{ij} = \xi\Big(-8E^i_{\rm ext}B^j_{\rm ext} + 14B^i_{\rm ext}E^j_{\rm ext} + 14\delta^{ij}(\mathbf{E}_{\rm ext}\cdot\mathbf{B}_{\rm ext})\Big)$$

BIREFRINGENT SIGNAL

Coulomb field:

$$\int d^3r\, e^{i\Delta\mathbf{k}\cdot\mathbf{r}}\, \frac{Q\mathbf{e}_r}{4\pi\mathbf{r}^2} = iQ\frac{\Delta\mathbf{k}}{(\Delta\mathbf{k})^2}$$

Forward scattering amplitude:

$$\mathfrak{U}^{\perp} = 6i\xi\frac{Q}{(\Delta\mathbf{k})^2}\frac{\omega^2}{4\pi}\left[(\mathbf{e}_{\text{out}}\cdot\mathbf{B}_{\text{ext}})(\mathbf{e}_{\text{out}}\cdot\Delta\mathbf{k}) - (\mathbf{e}_{\text{in}}\cdot\mathbf{B}_{\text{ext}})(\mathbf{e}_{\text{in}}\cdot\Delta\mathbf{k})\right]$$

$$B_{\text{ext}} = 10^6 T \quad , \quad I \approx 10^{22} W/cm^2 \quad , \quad \omega = 24 keV$$

$$\frac{d\sigma}{d\Omega} = \left|\frac{\alpha_{\text{QED}}^2}{15\pi}\frac{qB_{\text{ext}}}{m^2}\frac{\omega^2}{m^2}\frac{Z}{|\Delta\mathbf{k}|}\right|^2 \sim 10^{-25}\,\frac{Z^2}{(\Delta\mathbf{k})^2}$$

Momentum transfer $\Delta\mathbf{k} = \mathcal{O}(\text{eV})$

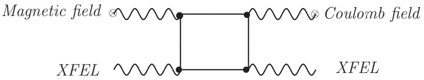
Peak in forward direction $\vartheta = \mathcal{O}(\text{mrad})$

Large interaction volume $\mathcal{O}(1/|\Delta\mathbf{k}|)$

Many $N = \mathcal{O}(10^8)$ nuclei $\rightarrow$ coherent superposition $Z_{\text{eff}} = NZ$

BACKGROUND SCATTERING PROCESSES

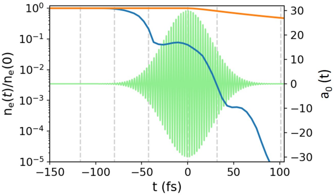
Problem: residual electrons!

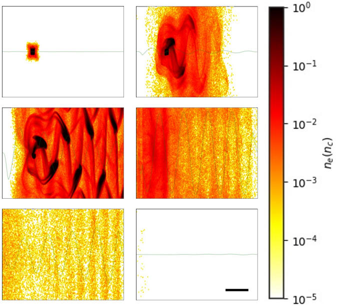


$t = -117fs$, $t = -80fs$

$t = -43fs$, $t = 32fs$

$t = 101fs$





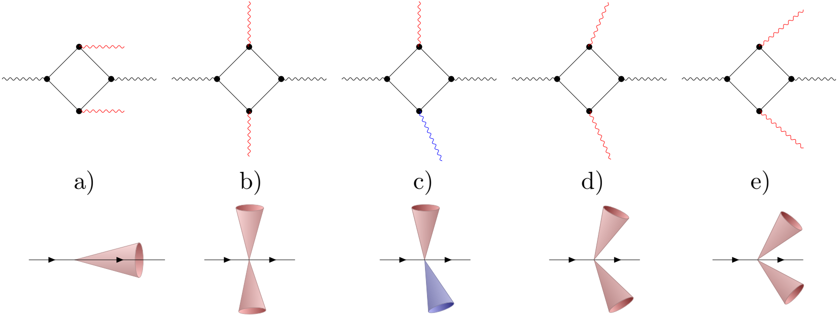
N. Ahmadiniaz, M. Bussmann, T.E. Cowan, A. Debus, T. Kluge, R. Schützhold, Phys. Rev. D Lett. **104**, 011902 (2021)

N. Ahmadiniaz, T. E. Cowan. M. Ding, M. A. Lopez-Lopez, R. Sauerbrey, R. Shaisultanov, R. Schützhold, arXiv: 2212.03350 [hep-th]

X-RAY+OPTICAL LASER

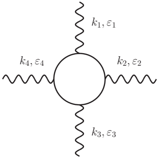
N. Ahmadiniaz, T. E. Cowan, J. Grenzer, S. Franchino-Viñas, A. Laso Garcia, M. Smid, T. Toncian, M. A. Trejo and R. Schützhold

arXiv: 2208.141215 [physics-optics]



- Priority access project 5438, March 2024
- Combining X-ray with Relax lasers, Autumn 2024.
- Full experiment 2025 (Dark field scenario, Karbstein et al, PRL **129** 061802 (2022)).

FOUR-PHOTON AMPLITUDE



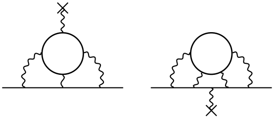
- General four-photon from Euler-Heisenberg $\Rightarrow$ only for small ω
- Four-photon amplitude $\Rightarrow$ much more nontrivial

N. Ahmadiniaz, C. Lopez-Arcos, M. Lopez-Lopez, C. Schubert
Nucl. Phys. **B 991** (2023) 116216
Nucl. Phys. **B 991** (2023) 116217.

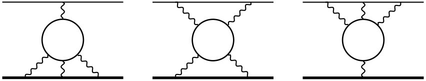
REMINDER

* 1950 R. Karplus and M. Neuman $\rightarrow$ arbitrary kinematics (on-shell)
* 1964, 1965 B. De Tollis $\rightarrow$ more compact representation (on-shell)
* 1971 V. Costantini, B. De Tollis and G. Pistoni $\rightarrow$ two photons on-shell and two off-shell $\rightarrow$ the most nontrivial case to date.
* Our aim is to compute the same amplitude fully off-shell with one and two in low-energy.

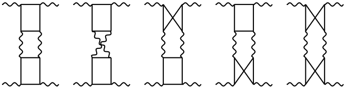
WHY OFF-SHELL 4-PHOTON AMPLITUDE?



Three and four-loop $g-2$



Three-loop Lamb shift



Three-loop to light-by-light

WORLDLINE FORMALISM AND N-PHOTON AMPLITUDE

- 1988-1991 Z. Bern and D. A. Kosower: one loop gluon amplitudes from string theory.
- 1992 M. J. Strassler: photon and gluon amplitudes from worldline path integrals. Worldline path integrals were invented by Feynman in 1950-1951.

One-loop effective action for **off-shell** scalar QED in the worldline representation

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(T)=x(0)} \mathcal{D}x \; e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ie\dot{x}\cdot A(x)\right)} \tag{1}$$

For the case of photon-amplitudes the background field $A(x)$ must be taken as a sum of plane waves

$$A_\mu(x) = \sum_{i=1}^{N} \varepsilon_{i\mu} \; e^{ik_i\cdot x},$$

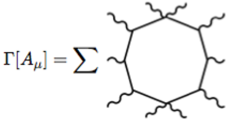
WORLDLINE FORMALISM FOR PHOTON AMPLITUDES

By a Gaussian integration and doing some integration by parts (to remove all $\ddot{G}$'s), we can obtain a master formula for the N-photon amplitude

$$\begin{aligned}\Gamma[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] &= (-ie)^N(2\pi)^D\delta\left(\sum_{i=1}^{N}k_i\right)\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\\ &\quad\times\prod_{k=1}^{N}\int_0^T d\tau_k Q_N(\dot{G}_{ij})\exp\left[\frac{1}{2}\sum_{i,j=1}^{N}G_{ij}k_i\cdot k_j\right]\end{aligned} \tag{2}$$

where G_{ij} is the worldline Green's function

$$\begin{aligned}G_{ij}\equiv G(\tau_i-\tau_j) &= |\tau_i-\tau_j|-\frac{(\tau_i-\tau_j)^2}{T}\\ \dot{G}_{ij} &= \mathrm{sgn}(\tau_i-\tau_j)-2\frac{\tau_i-\tau_j}{T}\end{aligned} \tag{3}$$



ADVANTAGES OF THE WORLDLINE FORMALISM

- We have the sum of many Feynman diagrams in a single expression.
- Permutation invariance is maintained through-out.
- We have a replacement rule to obtain the spinor amplitude from the scalar one.
- It is possible to express the amplitude such that gauge invariance is manifest at the integrand level.
- For $N = 4$ the IBP procedure also eliminates the spurious UV divergences.

FOUR-PHOTON AMPLITUDE

The integral form of the four-photon amplitude for scalar QED can be written as

$$\Gamma_{\text{scalar}}[k_1,\varepsilon;\cdots;k_4,\varepsilon_4] = (-ie)^4 \int_0^T \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\ \mathrm{e}^{-m^2 T} \int_0^T \prod_{i=1}^4 d\tau_i\, Q_4(\dot{G}_{ij}) \exp\Big[\sum_{i,j=1}^4 \frac{1}{2} G_{ij} k_i\cdot k_j\Big] \qquad (4)$$

$$\begin{aligned} Q_4 &= Q_4^4 + Q_4^3 + Q_4^2 + Q_4^{22} \\ Q_4^4 &= \dot{G}(1234) + 2\,\text{perms} \\ Q_4^3 &= \dot{G}(123)T(4) + 3\,\text{perms} \\ Q_4^2 &= \dot{G}(12)T(34) + 5\,\text{perms} \\ Q_4^{22} &= \dot{G}(12)\dot{G}(34) + 2\,\text{perms} \end{aligned} \qquad (5)$$

$$\dot{G}(i_1 i_2\cdots i_n) := \dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_n i_1}\Big(\frac{1}{2}\Big)^{\delta_{n,2}} \mathrm{tr}(f_{i_1} f_{i_2}\cdots f_{i_n}) \qquad (6)$$

$$T(i) = \sum_{r\neq i} \dot{G}_{ir}\varepsilon_i\cdot r \quad , \quad f_{\mu\nu} = k_\mu\varepsilon_\nu - k_\nu\varepsilon_\mu$$

$$T(ij) = \sum_{r\neq i, s\neq j} \dot{G}_{ir}\varepsilon_i\cdot k_r \dot{G}_{js}\varepsilon_j\cdot k_s + \frac{1}{2}\dot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\Big[\sum_{r\neq i,j}\dot{G}_{ir}k_i\cdot k_r - \sum_{s\neq j,i}\dot{G}_{js}k_j\cdot k_s\Big]$$

FOUR-PHOTON AMPLITUDE, TWO LOW ENERGY PHOTONS

The four-photon amplitude for spinor QED with two photons of low energy is

$$\Gamma_{\mathrm{spin}(34)}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] = -\frac{2e^4}{(4\pi)^{\frac{D}{2}}}\hat{\Gamma}_{\mathrm{spin}(34)}(\tilde{Q}_4) \quad , \quad f_{\mu\nu} = k_\mu\varepsilon_\nu - k_\nu\varepsilon_\mu$$

where $\hat{\Gamma}_{\mathrm{spin}(34)}$ is composed of the sum of the following expressions $(Z_n(i_1 i_2\cdots i_n) = \mathrm{tr}(f_{i_1} f_{i_2}\cdots f_{i_n}))$

$$\hat{\Gamma}^4_{\mathrm{spin}(34)}(1234) = -\frac{4}{3}Z_4(1234)\,(Y_{41}+2Y_{42}) \quad (\mathrm{two}-\mathrm{more})$$

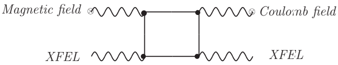
$$\hat{\Gamma}^3_{\mathrm{spin}(34)}(123;4) = \frac{2}{9}Z_3(123)k_2\cdot f_4\cdot k_1\,(Y_{51}-Y_{52}-12Y_{53}) \quad (\mathrm{on}-\mathrm{more})$$

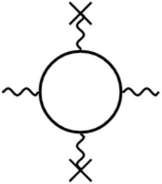
$$\hat{\Gamma}^2_{\mathrm{spin}(34)}(12;34) = \frac{1}{3}Z_2(12)\left[\frac{2}{15}Y_{51}\,(k_1\cdot f_4\cdot f_3\cdot k_1 + k_2\cdot f_4\cdot f_3\cdot k_2) + \frac{2}{15}\,(Y_{51}-30Y_{53})\right.$$
$$\left.\times\,(k_1\cdot f_4\cdot f_3\cdot k_2 + k_2\cdot f_4\cdot f_3\cdot k_1) + \frac{4}{3}\,(Y_{63}-4Y_{64})\,k_2\cdot f_4\cdot k_1 k_2\cdot f_3\cdot k_1\right] \quad (\mathrm{four}-\mathrm{more})$$

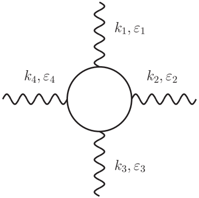
$$\hat{\Gamma}^{22}_{\mathrm{spin}(34)}(12,34) = \frac{8}{3}Z_2(12)Z_2(34)Y_{41} \quad (\mathrm{two}-\mathrm{more})$$

$$Y_{nl} \equiv \frac{\Gamma(n-\frac{D}{2}-l)}{m^{2n-D}}\,\frac{d^l}{dx_i^l}\,{}_2F_1(1,n-\frac{D}{2}-l;\frac{3}{2};\frac{k_1\cdot k_2}{4m^2})$$

SUMMARY







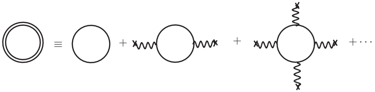
- Coulomb assisted Vacuum birefringence
- Fully laser based Vacuum birefringence
- Generalized four-photon amplitude beyond Euler-Heisenberg

Thank you for your attention

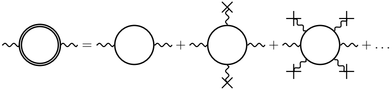
Back up slides

LOW-FIELD LIMIT

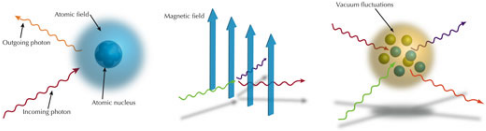
Euler-Heisenberg Lagrangian:



Vacuum polarization:



photon-photon interaction:



Adopted from: ELI ERIC

EXPERIMENTAL REALIZATION OF LIGHT-BY-LIGHT

- LbL scattering (indirectly) via electron loop ($g-2$), D. Hanneke et al, PRL **100**, 120801 (2008)).
- LbL via Delbrück scattering, at fixed photon energy of 7 GeV, S. Z. Akhmadaliev et al, PRC **58**, 2844 (1998).
- photon splitting: A photon splits into two photons in external fields, $(0.1-0.5$ GeV), S. Z. Akhmadaliev et al, PRL **89**, 061802 (2002).
- ATLAS experiment (Nature Phys.**13**, 852 (2017)):
  An alternative way by which LbL interactions can be studied is by using relativistic heavy-ion collisions. The electromagnetic field strengths of relativistic ions scale with the proton number (Z). For a lead (Pb, $Z=82$) nucleus the field can be up to $10^{25}V/m$, much larger than E_{cr}.

FOUR-PHOTON AMPLITUDE, TWO LOW ENERGY PHOTONS

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where $\hat{\Gamma}_{\text{spin}(34)}$ is composed of the sum of the following expressions $(Z_n(i_1 i_2 \cdots i_n) = \text{tr}(f_{i_1} f_{i_2} \cdots f_{i_n}))$

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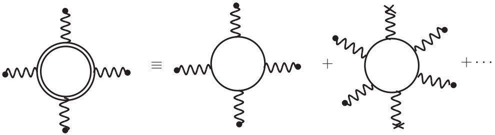
$$\hat{\Gamma}^3_{\text{spin}(34)}(123;4) = \frac{2}{9} Z_3(123) k_2 \cdot f_4 \cdot k_1\,(Y_{51} - Y_{52} - 12Y_{53}) \quad (\text{on} - \text{more})$$

$$\hat{\Gamma}^2_{\text{spin}(34)}(12;34) = \frac{1}{3} Z_2(12) \Bigg[\frac{2}{15} Y_{51}\,(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2) + \frac{2}{15}(Y_{51} - 30Y_{53})$$
$$\times\,(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1) + \frac{4}{3}(Y_{63} - 4Y_{64})\,k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1 \Bigg] \quad (\text{four} - \text{more})$$

$$\hat{\Gamma}^{22}_{\text{spin}(34)}(12,34) = \frac{8}{3} Z_2(12) Z_2(34) Y_{41} \quad (\text{two} - \text{more}) \quad , \quad \text{f}_{\mu\nu} = \text{k}_\mu \varepsilon_\nu - \text{k}_\nu \varepsilon_\mu$$

$$Y_{nl} \equiv \frac{\Gamma(n - \frac{D}{2} - l)}{m^{2n-D}} \frac{d^l}{dx_i^l}\, {}_2F_1(1, n - \frac{D}{2} - l; \frac{3}{2}; \frac{k_1 \cdot k_2}{4m^2})$$

LOW-ENERGY FOUR-PHOTON IN A MAGNETIC FIELD



$$\Gamma_{\mathrm{scal}}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] = (-ie)^4 \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\, \mathrm{e}^{-m^2 T}$$

$$\det{}^{-1/2}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right] \quad \prod_{i=1}^{4}\int_0^1 du_i Q(\mathcal{G}_{ij})\, \mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^{4} k_i\cdot\mathcal{G}_{ij}\cdot k_j} \tag{7}$$

$$G_B(\tau_i,\tau_j) \quad\to\quad \mathcal{G}_B(\tau_i,\tau_j) \equiv \frac{T}{1\mathcal{Z}^2}\Big(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\,\mathrm{e}^{-i\mathcal{Z}\dot{G}_{Bij}+i\mathcal{Z}\dot{G}_{Bij}} - 1\Big)$$

$$G_F(\tau_i,\tau_j) \quad\to\quad \mathcal{G}_F(\tau_i,\tau_j) \equiv G_{Fij}\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{Bij}}}{\cos\mathcal{Z}} \quad , \quad \mathcal{Z}_{\mu\nu} = eTF_{\mu\nu} \tag{8}$$

N.A. et al (in preparation)

- T. Heinzl et al, Opt. Commun.267 (2006) 318.
- H.-P. Schlenvoigt et al, Phys. Scr.91 (2016) 023010.
- F. Karbstein et al; arXiv: 2105.13869 [hep-th]
- K. S. Schulze et al. Phys. Rev. Research 4 (2022) 013220
- https://www.ts.infn.it/physics/experiments/pvlas/
- https://www.xfel.eu/index-ger.html
- N. A, T. E. Cowan, J. Grenzer, S. Franchino-Viñas, A. Laso Garcia, M. Smid, T. Toncian, M. A. Trejo and R. Schützhold
  arXiv: 2208.141215 [physics-optics]

HISTORY AND INTRODUCTION

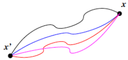
In 1948, Feynman developed the path integral approach to non-relativistic quantum mechanics (based on earlier work by Wentzel and Dirac). Two years later, he started his famous series of papers that laid the foundations of relativistic quantum field theory (essentially quantum electrodynamics at the time) and introduced Feynman diagrams. However, at the same time he also developed a representation of the QED S-matrix in terms of relativistic particle path integrals.

Why worldline formalism?

- No need to compute momentum integrals and Dirac traces.
- Worldline formalism works well for massive particles (on- and off-shell) not even at tree-level but at loop order too.

The difference between open line and loop (purely bosonic):

- Dirichlet boundary conditions (topology of a line)



$$\langle x|e^{-HT}|x'\rangle = \int_{x(0)=x'}^{x(T)=x} Dx(\tau)\, e^{-S[x,G]}$$

- Periodic boundary conditions (topology of a closed line)

$$\int_{x(0)=x(T)} Dx(\tau) e^{-S[x,G]}$$

Free-propagator:
Free scalar propagator that is the Green's function for the Klien-Gordon equation:

$$D_0^{x,x'} = \langle 0|T\phi(x)\phi(x')|0\rangle = \langle x|\frac{1}{-\Box + m^2}|x'\rangle \;, \quad \Box = \sum_{i=1}^{4}\frac{\partial^2}{\partial x_i^2}$$

We exponentiate the denominator using a Schwinger proper-time parameter T. This gives

$$D_0^{xx'} = \int_0^\infty dT\, e^{-m^2 T}\langle x|\exp\left[-T(-\Box)\right]|x'\rangle = \int_0^\infty dT\, e^{-m^2 T}\int_{x(0)=x'}^{x(T)=x}\mathcal{D}x\, e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2}$$

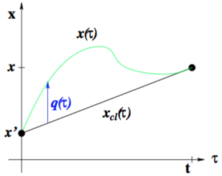
This is the *worldline path integral representation* of the relativistic propagator of a scalar particle in euclidean spacetime from x' to x.

Having found this path integral, let us calculate it, as a consistency check. First, let us perform a change of variables from $x(\tau)$ to $q(\tau)$, defined by

$$x^\mu(\tau) = x^\mu_{\rm cl} + q^\mu(\tau) = [x^\mu + \frac{\tau}{T}(x^\mu - x'^\mu)] + q^\mu(\tau)$$

$x_{\rm cl}$ is the free classical trajectory fulfilling the path integral boundary conditions, $x_{\rm cl}(0) = x'$ and $x_{\rm cl}(T) = x$ and $q(\tau)$ is fluctuating quantum variable around the classical one, fulfilling the Dirichlet boundary condition (DBC), $q(0) = q(\tau) = 0$. By this change of variable, the propagator reads as

$$D_0^{xx'} = \int_0^\infty dT\, e^{-m^2 T}\, e^{-\frac{(x-x')^2}{4T}} \int_{DBC} \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{q}^2}$$



$\Rightarrow$ Fourier transform $\Rightarrow$ familiar momentum space representation

$$D_0^{pp'} = \int d^D x e^{ip\cdot x} \int d^D x' e^{ip'\cdot x'} D_0^{xx'}$$

$$= \boxed{(2\pi)^D \delta^D(p + p') \frac{1}{p^2 + m^2}}$$

COUPLING TO ELECTROMAGNETIC FIELD

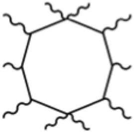
To get the the "full" or "complete" propagator for a scalar particle, that interacts with background field $A(x)$ continuously while propagating from x' to x

$$D^{xx'}[A] = \int_0^\infty dT\, e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x(\tau))\right)}$$

Effective action: The effective action encodes the nonlinear properties of a system due to quantum fluctuations, analogously to how the thermodynamic partition function encodes the effects of thermal fluctuations.

$$\Gamma_{\text{scal}}[A] = \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x(\tau))\right)}$$

Note that we now have a dT/T, and that the path integration is over closed loops; those trajectories can therefore belong only to virtual particles, not to real ones. The effective action contains the quantum effects caused by the presence of such particles in the vacuum for the background field. In particular, it causes electrodynamics to become a nonlinear theory at the one-loop level, where photons can interact with each other in an indirect fashion.

$$\Gamma[A_\mu] = \sum$$ 

After the following decomposition (to fix the average position of the loop)

$$x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$$

$$\int \mathcal{D}x(\tau) = \int d^D x_0 \int \mathcal{D}y(\tau) \quad , \quad x_0^\mu \equiv \frac{1}{T}\int_0^T d\tau x^\mu(\tau) \tag{9}$$

The remaining $y(\tau)$ path integral is performed using the Wick contraction rule

$$\langle y^\mu(\tau_i) y^\nu(\tau_j)\rangle = -\delta^{\mu\nu} G_B(\tau_i,\tau_j) \quad , \quad G_B(\tau_i,\tau_j) \equiv G_{Bij} = |\tau_i - \tau_j| - \frac{(\tau_i-\tau_j)^2}{T} \tag{10}$$

The free Gwaussian path integral gives

$$\int \mathcal{D}\, \mathrm{e}^{-\int_0^T d\tau \frac{1}{4}\dot{y}^2} = (4\pi T)^{-\frac{D}{2}} \quad , \quad D = \text{spacetime dimension} \tag{11}$$

Now if we specialize the background $A(x)$, which so far was an arbitrary Maxwell field, to a sum of N plane waves,

$$A^\mu(x) = \sum_{i=1}^N \varepsilon_i^\mu \, e^{ik_i\cdot x}$$

After expanding the interaction term we get

$$\Gamma_{\text{scal}}[A] = (-ie)^N \int_0^\infty \frac{dT}{T}\, e^{-m^2 T}(4\pi T)^{-\frac{D}{2}} \left\langle V_{\text{scal}}^\gamma[k_1,\varepsilon_1]\cdots V_{\text{scal}}^\gamma[k_N,\varepsilon_N]\right\rangle$$

$$\Rightarrow V_{\text{scal}}^\gamma[k,\varepsilon] \equiv \int_0^T d\tau \varepsilon\cdot\dot{x}\, e^{ik\cdot x}$$

At this stage the zero-mode integration can be performed

$$\int d^D x_0 \prod_{i=1}^{N} e^{ik_i \cdot x_0} = (2\pi)^D \delta^D (\sum_{i=1}^{N} k_i)$$

After some mathematical manipulations one get the following master formula which is known as **Bern-Kosower** master formula for external photons

$$\begin{aligned} \Gamma_{\rm scal}[k_1, \varepsilon_1; \cdots ; k_N, \varepsilon_N] &= (-ie)^N (2\pi)^D \delta^D (\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^{N} \int_0^T d\tau_i \\ &\times \exp \left\{ \sum_{i,j=1}^{N} \left[\frac{1}{2} G_{Bij} (k_i \cdot k_j) - i \dot{G}_{Bij} (\varepsilon_i \cdot k_j) + \frac{1}{2} \ddot{G}_{Bij} (\varepsilon_i \cdot \varepsilon_j) \right] \right\} \Big|_{\rm lin(\varepsilon_1 \cdots \varepsilon_N)} \end{aligned}$$

where

$$\begin{aligned} \dot{G}_{Bij} &= \frac{dG_{Bij}}{d\tau_i} = \mathrm{sign}(\tau_i - \tau_j) - \frac{2(\tau_i - \tau_j)}{T} \\ \ddot{G}_{Bij} &= \frac{d^2 G_{Bij}}{d\tau_i^2} = 2\delta(\tau_i - \tau_j) - \frac{2}{T} \end{aligned} \quad (12)$$