Elasticity, Geometry and Fractons

Towards hydrodynamics with internal d.o. f.

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Görlitz, 05.12.2023

Thermalisation of many-body systems

Castelnovo

Bravyi et al.

Thermalisation is the process of physical bodies reaching thermal equilibrium through mutual interaction. Although it is natural for physical systems to reach equilibrium not all quantum states undergo thermalisation. Mechanisms discovered to prevent thermalisation:

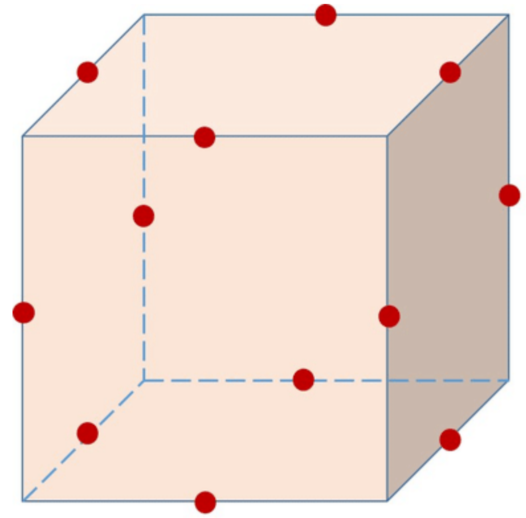
- Kinematically constrained dynamics
- Many body localisation
- Integrability
- Quantum scars

Fractons are excitations that cannot move in isolation. As we will see an example of such constrained dynamics is obtained if we impose charge and dipole conservation. This is an example of a glass-like system.

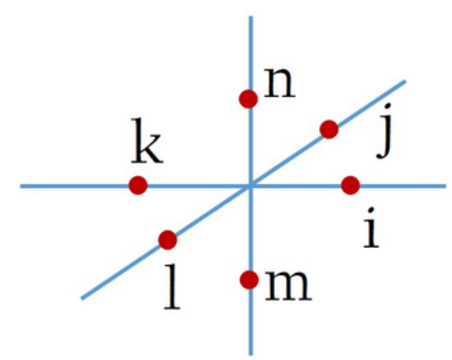
Gapped fractons

Fracton phases constitute a new class of quantum state of matter. They are characterized by excitations that exhibit restricted mobility, being either immobile under local Hamiltonian dynamics, or mobile only in certain directions.

X-cube model - represented by spins placed on the links of a cubic lattice and is given by the sum of a 12-spin Pauli-x operator at each cube and planar four-spin Pauli-z operators at each vertex.

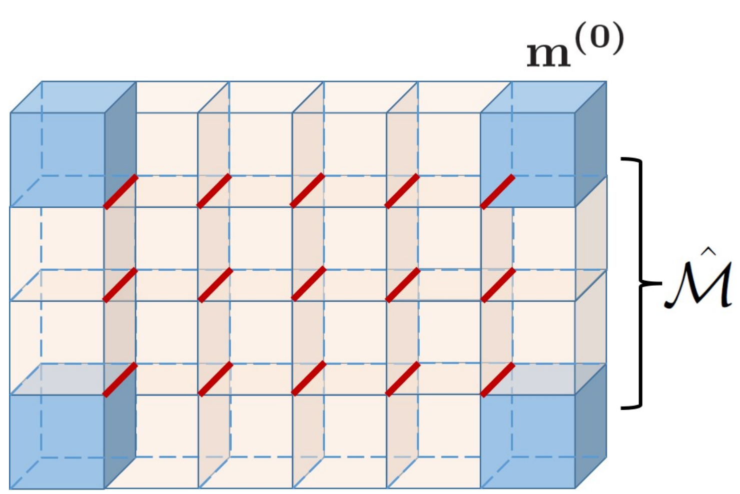


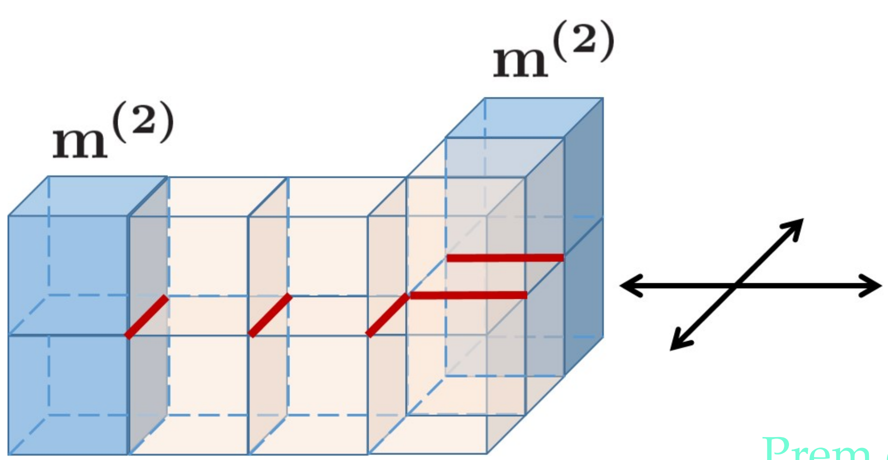
$$A_c = \prod_{j \in \partial c} \sigma_x^j$$



$$B_v^{(xy)} = \sigma_z^i \sigma_z^j \sigma_z^k \sigma_z^l$$
$$B_v^{(yz)} = \sigma_z^j \sigma_z^n \sigma_z^l \sigma_z^m$$
$$B_v^{(xz)} = \sigma_z^i \sigma_z^n \sigma_z^k \sigma_z^m$$

Haah



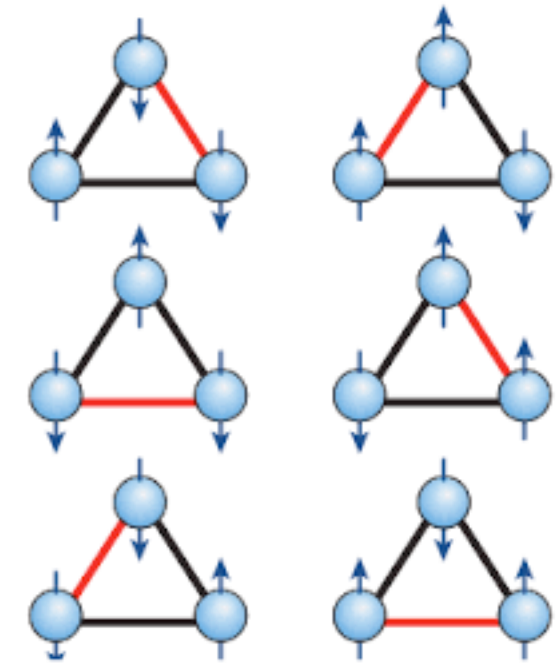


Prem et al.

Topological excitations of the X-Cube model are immobile fractons at the corners and composite topological excitations mobile along two-dimensional submanifolds.

Fractons in spin liquids

Pretko



In addition to gapped fractons gapless fractons were considered by Pretko, motivated by spin liquids. A quantum spin liquid is a state of matter in which the spins are highly entangled and don't order, even at zero temperature. In the simplest case the low-energy, long-wavelength physics of spin liquids is given by the electromagnetic energy. Pretko conjectured a generalisation of this idea introducing tensor gauge fields

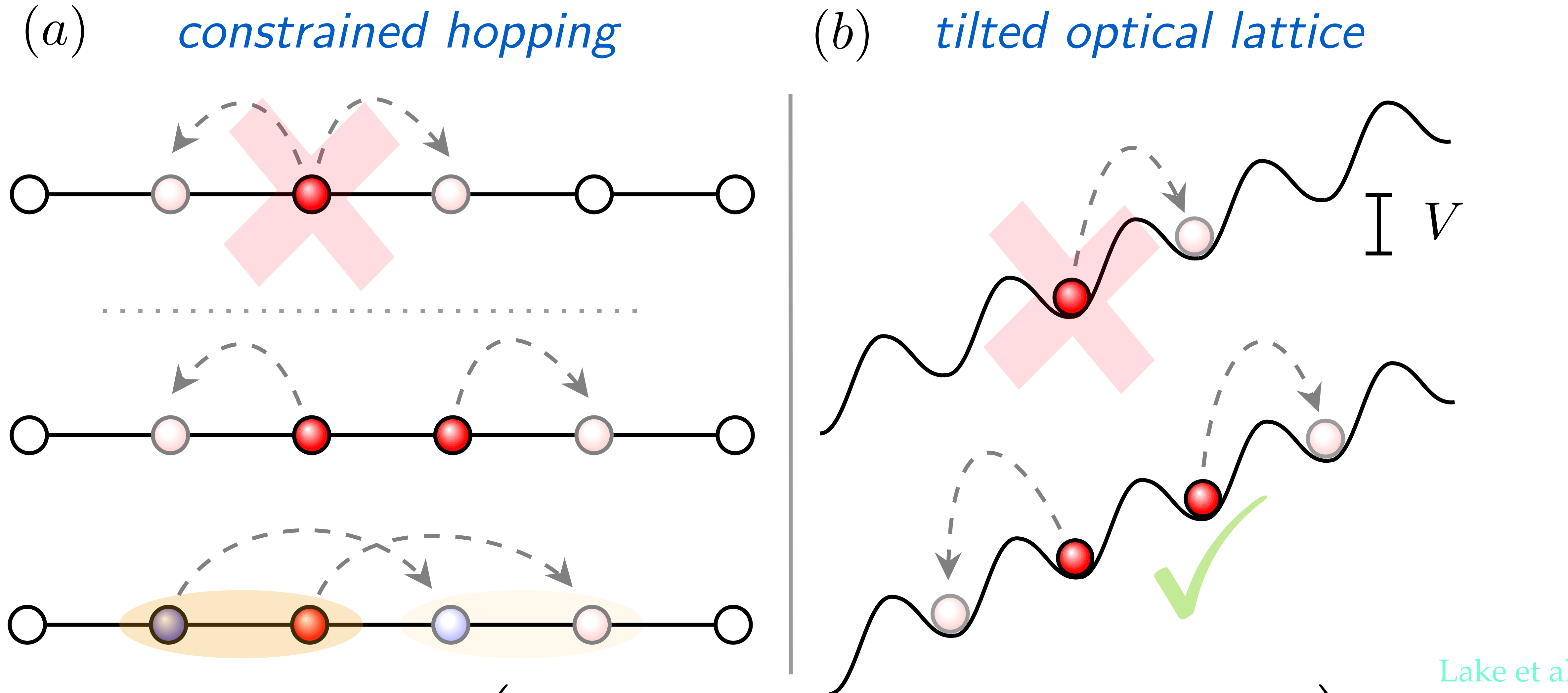
$$B^i = \epsilon^{kl}\partial_k A^i_l\,, \qquad E^i_j = \epsilon^i{}_k(-\partial_0 A^k_j + \partial_j\partial_k\phi)$$

An important observation is that if we couple sources to this theory, both the charge and the dipole moment will be conserved

$$Q = \int d^2x\rho = \text{const}\,, \qquad D^i = \int d^2x\rho x^i = \text{const}$$

This implies that the charges cannot move freely but are constrained in analogy to the X-cube model. Quite remarkably symmetric tensor gauge theories emerge in the dual formulation of elasticity.

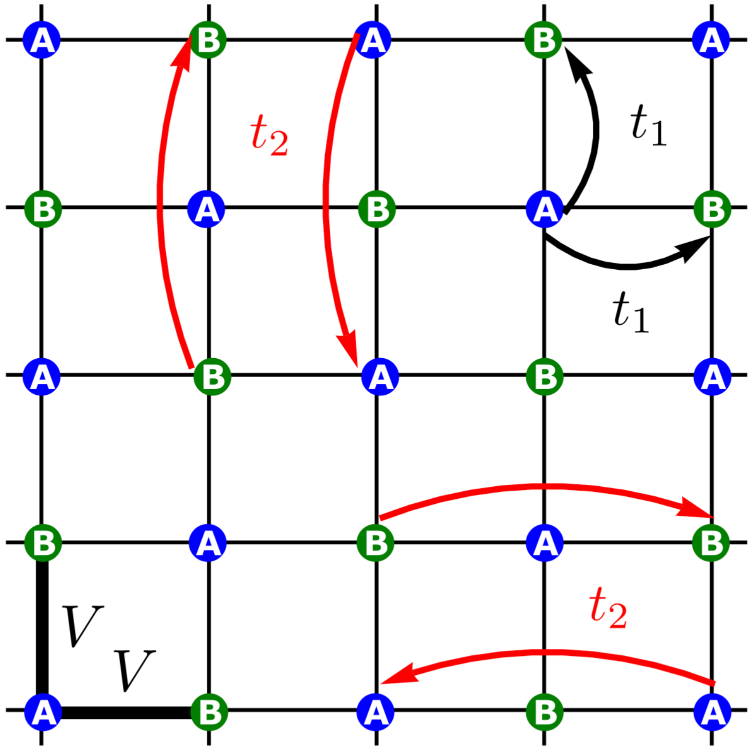
Bose-Hubbard realisation in 1d



$$H_{tilted} = \sum_i \left(-t b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i + \sum_i V i n_i + \frac{U}{2} n_i^2 \right)$$

The restricted kinematics of dipole conservation. An isolated boson cannot move, while two nearby bosons can move only by coordinated hopping in opposite directions. A boson and a hole (blue circle) can move freely in both directions. (b) Approximate dipole conservation can be engineered in tilted optical lattices with large tilt strength V . Energy conservation then forbids single bosons from hopping, while dipole-conserving hopping processes are allowed.

Bose-Hubbard realisation in 2d



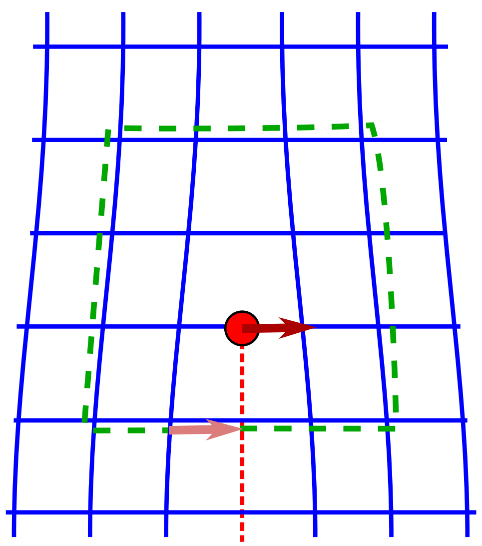
$$\hat{H} = -t_1 \sum_{\langle ij \rangle} \hat{b}_i^\dagger \hat{b}_j + \frac{V}{2} \sum_{\langle ij \rangle} \hat{n}_i \hat{n}_j$$

$$-t_2 \sum_{[ijkl]} \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l$$

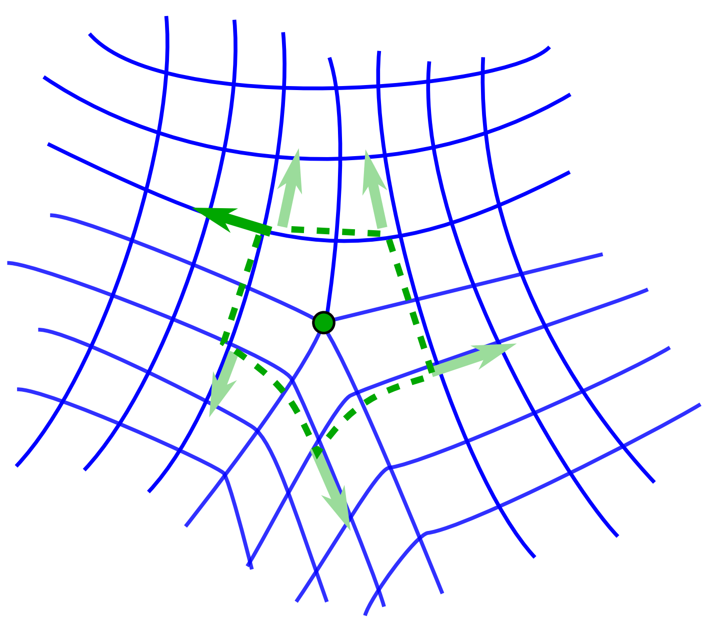
Giergiel et al.

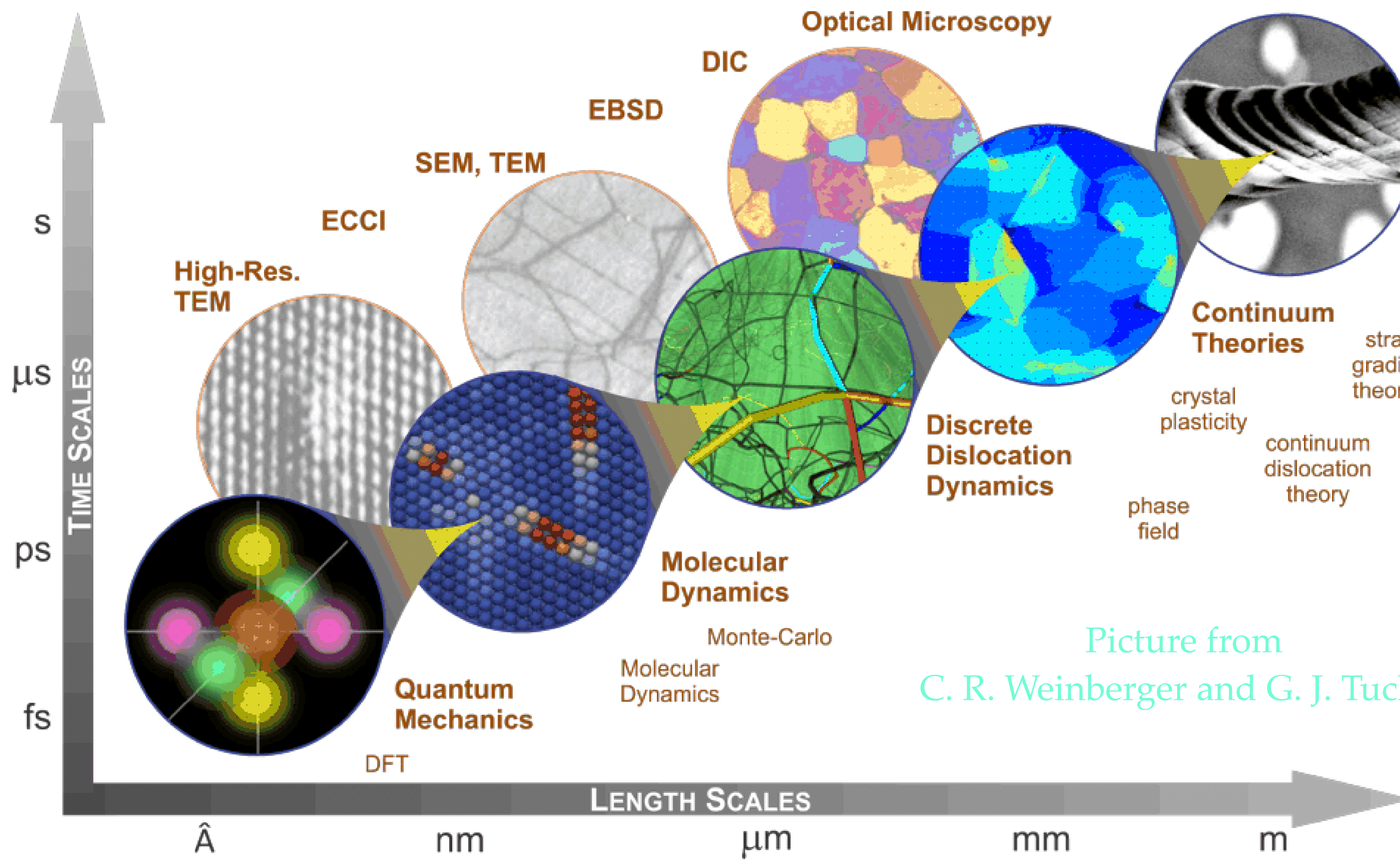
The first two terms of the Hamiltonian describe a familiar hard-core boson extended Bose-Hubbard model with t_1 and V being the tunneling amplitude and the nearest neighbor interaction strength respectively. The symbol $\langle ij \rangle$ indicate the sum over nearest neighbors on the lattice. The last term contains simultaneous tunneling processes of two particles, dubbed the ring-exchange interaction. The ring-exchange interaction we are considering is a combination of two second neighbor hoppings along 1×2 and 2×1 plaquettes which preserves the center of mass of two particles and the number of particles in bipartite sublattices. The symbol $[ijkl]$ denotes the summation over all possible plaquettes. We considering strongly repulsive limit $V \gg t_1, t_2$, where the ground state realizes the checkerboard charge density wave (CDW) order.

Defect dynamics in crystals



A dislocation is a half-line insertion, indicated by the dashed red line. Its topological charge is a Burgers vector, indicated by an arrow. This charge can be picked up by traversing a contour around the defect as indicated by the dotted line. The topological defects associated with rotational order are called disclinations.

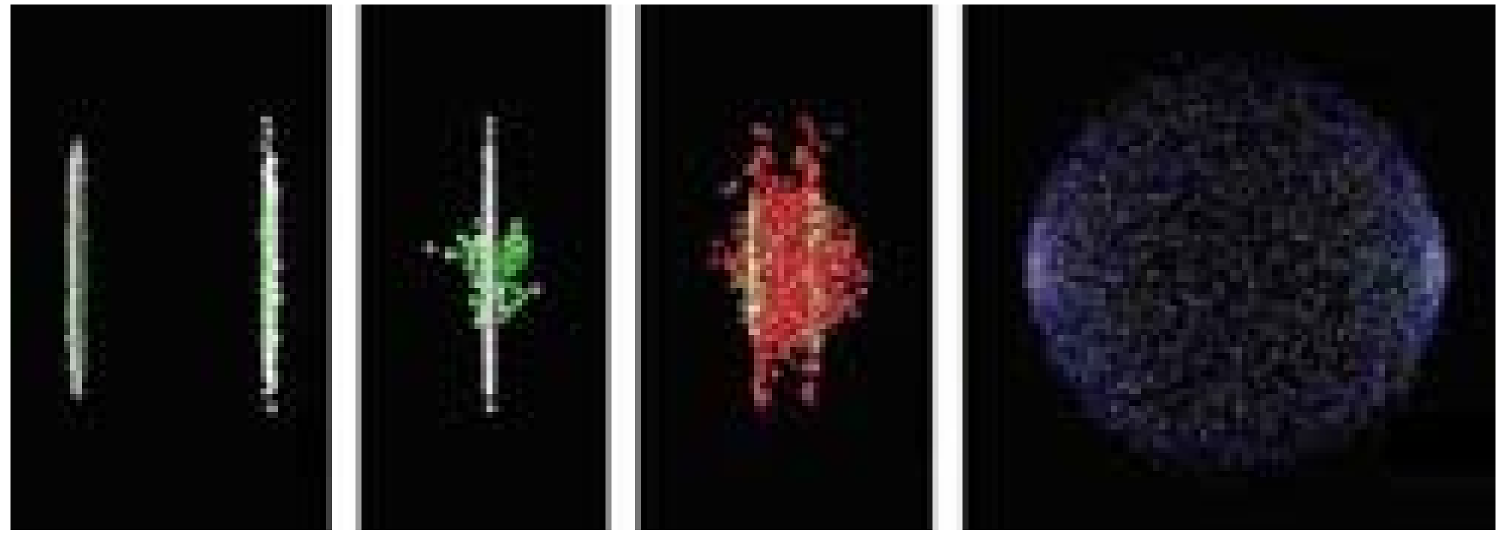




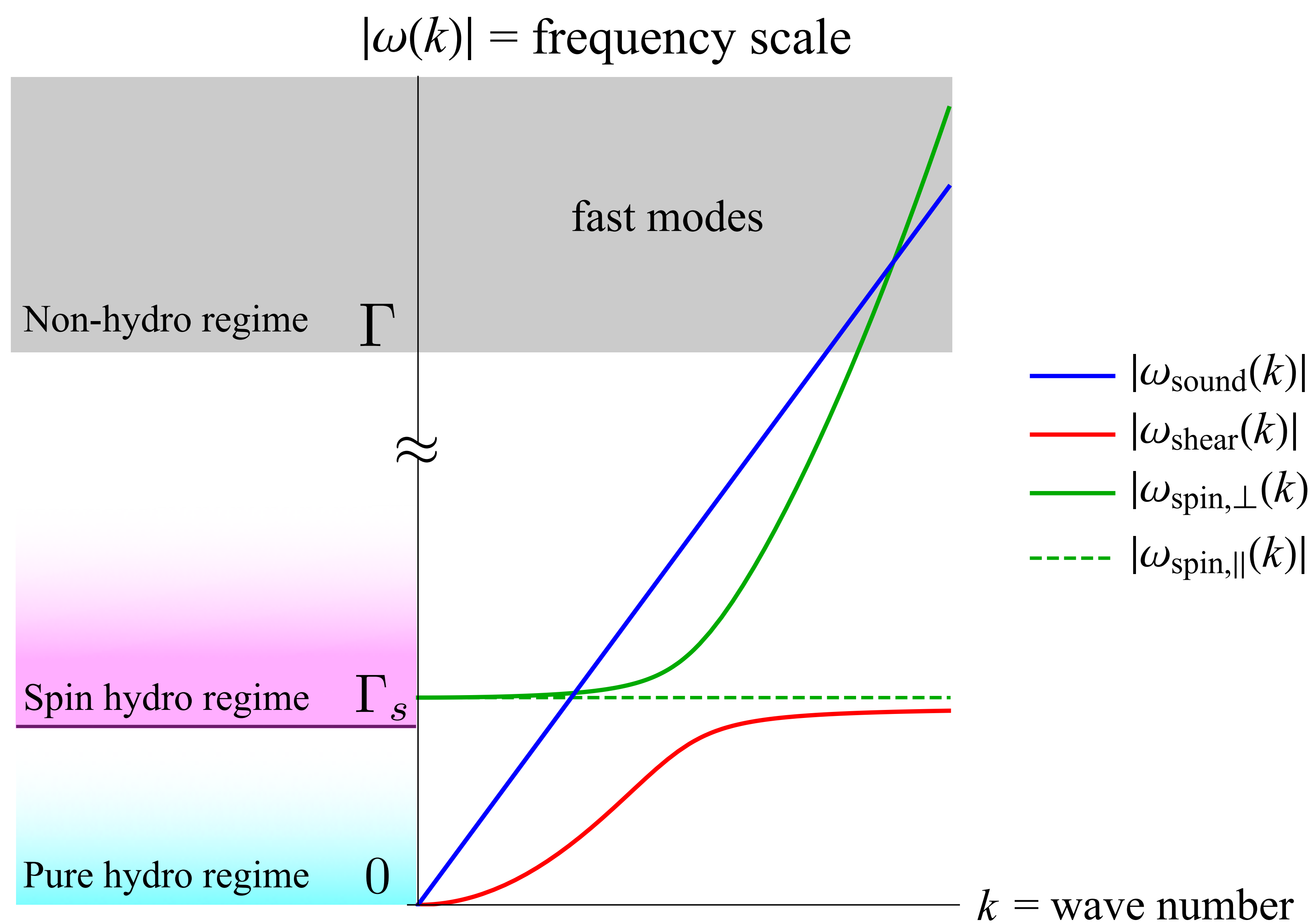
Picture from
C. R. Weinberger and G. J. Tucker

Spin hydro, Micropolar fluids

In non-central heavy-ion collisions, the generated matter is both hot and dense and carries significant initial orbital angular momentum. This momentum results in the global polarization of hadrons in the final state. Such polarization can be detected through the parity-violating weak decays of hyperons. The STAR experiment has recorded a non-zero level of global polarization. With the availability of an extensive new dataset, there is now the opportunity to assess the polarization of multistrange hyperons. These measurements could provide valuable insights for hydrodynamic analysis of the system.



Hongo et al.



From solids to fluids: hydrodynamics - theory of conserved quantities

This conservation of particle number is expressed in hydrodynamics as conservation of mass, by the continuity equation

$$\partial_t \rho + \partial_i(\rho u_i) = 0$$

Another equation is the equation of motion of a fluid element. This equation can be written as a momentum conservation equation.

$$\partial_t(\rho u_i) + \partial_j T_{ij} = 0 \qquad T_{ij} = p\delta_{ij} + \rho u_i u_j$$

We are still one equation short to have a complete system. We add entropy conservation equation, which can be expressed as energy conservation using thermodynamics

$$\partial_t \left(\varepsilon + \frac{\rho u^2}{2} \right) + \partial_i \left[\left(w + \frac{\rho u^2}{2} \right) u_i \right]$$

Rewriting we get the Euler's equation

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla p$$

Superfluids

A relativistic superfluid can be thought of as a fluid charged under a spontaneously broken U(1) symmetry. In the grand canonical ensemble the thermodynamic variables characterizing normal, charged fluids are the temperature and chemical potential, and their conjugate variables entropy and total charge density. Once the U(1) symmetry is spontaneously broken then the goldstone boson provides for another degree of freedom. The equations for the slow modes can be written as

$$\partial_t \phi + F(\phi, \nabla\phi, ...) = 0.$$

We assume that there exists an equilibrium solution for which F vanishes. The evolution for fluctuations can be deduced from a microscopic theory for the Hamilton's equations (Josephson relation)

$$\partial_t \phi = -v_i \partial_i \phi - \mu.$$

Hydrodynamics of crystals

Josephson relation is not a conservation equation. Therefore it does not conform to the usual form of hydrodynamic equations. In order to cure this one can take a derivative of both sides. Note, hover that this changes the counting of derivatives.

In analogy with superfluids equations governing crystals at finite temperature read

$$\partial_t \, \hat{c}^{\alpha}(\mathbf{r},t) + \nabla^a \hat{J}^a_{c^\alpha}(\mathbf{r},t) = 0$$

with the following densities and current densities,

$$(\hat{c}^{\alpha}) = (\hat{e}, \hat{\rho}, \hat{p}^b, \hat{u}^{bc}) \qquad \text{and} \qquad (\hat{J}^a_{c^\alpha}) = (\hat{J}^a_e, \hat{J}^a_\rho, \hat{J}^a_{p^b}, \hat{J}^a_{u^{bc}})\,.$$

Szamel

Gaspard, Mabillard

Symmetry

Hydrodynamics follows the evolution of conserved quantities. These are macroscopic expectation values of microscopic operators. They follow from the symmetries of the system. We explore systems with dipole symmetry. Monopole-dipole-momentum (MDMA) algebra

Monopole: Q Dipole: $Q_i, \qquad i = 1, \ldots, d.$

Momentum: $P_i, \qquad i = 1, \ldots, d.$

$$\{Q, Q\} = \{Q, Q_i\} = \{Q, P_i\} = 0.$$

$$\{P_i, Q_j\} = Q\delta_{ij}.$$

Heisenberg algebra.

Pena-Benitez

Transformation of fields: $\delta_{(\alpha,\beta,\gamma)}\Psi = \{\Psi, \alpha Q, \beta^i Q_i, \gamma^i P_i\}.$

Hydrodynamics from Poisson brackets

Landau

We express conserved charges in terms of densities

$$Q = \int d^d x q, \quad Q^i = \int d^d x x^i q, \quad P^i = \int d^d x p^i.$$

The starting point is the "current algebra", which one can derive from a microscopic theory (apart from entropy, which is postulated).

$$\{p_i(\mathbf{x}), \rho(\mathbf{y})\} = -\rho(\mathbf{x})\partial_{x^i}\delta(\mathbf{x}-\mathbf{y}),$$
$$\{p_i(\mathbf{x}), s(\mathbf{y})\} = -s(\mathbf{x})\partial_{x^i}\delta(\mathbf{x}-\mathbf{y}),$$
$$\{p_i(\mathbf{x}), p_j(\mathbf{y})\} = -[p_j(\mathbf{x})\partial_{x^i} + p_i(\mathbf{y})\partial_{x^j}]\delta(\mathbf{x}-\mathbf{y}).$$

The equations of motion follow from the Poisson brackets of the hydrodynamic variables with the Hamiltonian

$$\partial_t \rho = \{\rho, H\} = -\partial_i j^i,$$
$$\partial_t p_i = \{p_i, H\} = -\partial_k T_i^k.$$

Thermodynamics of dipole-conserving systems cannot be captured by the standard textbook treatment. It requires a modified approach that systematically incorporates kinematic constraints arising from the dipole conservation.

The internal energy density of a generic system in equilibrium is a function of the entropy and conserved charge densities, for example $\epsilon \equiv \epsilon(n, s, p_i)$. However, owing to the noncommutative structure of the algebra, dipole-conserving systems are not generic. In fact, the combination p_i/n has a shift symmetry under dipole transformations in analogy to a Nambu-Goldstone mode. Therefore, it could enter via the invariant combination $V_{ij} = \partial_i(n^{-1}p_j)$. It is then necessary to introduce a conjugate variable F_{ij} that, can be interpreted as a flux of dipoles.

Thus, we infer that different (constant) values of V_{ij} label distinct thermodynamic states. For such systems, we postulate that the first law of thermodynamics reads

$$d\epsilon = Tds + \mu dn + F_{ij} dV_{ij} \, .$$

Dipole-conserving fluids

Following the canonical paradigm of hydrodynamics, we consider the long-wavelength, near-equilibrium dynamics that is governed by the hydrodynamic variables, that is, the densities n, ϵ, p_i of the conserved charges. Macroscopic currents are then given by local expressions of the conserved densities organized in a systematic derivative expansion. In writing these constitutive relations a set of unknown parameters will emerge, known as transport coefficients, which are then constrained imposing the laws of thermodynamics, and Onsager relations.

Nonetheless, the non-standard structure the dipole symmetry introduces suggest we should consider the momentum of the system p_i to be of order $\mathcal{O}(p_i) \sim \mathcal{O}(\partial_i)^{-1}$, such that $V_{ij} \sim \mathcal{O}(\partial_i)^0$. Therefore, our derivative expansion is defined in terms of the order at which the equations of motion are truncated, e.g. we will refer to $n-$th order hydrodynamics if the set of differential equations is truncated as

$$\begin{aligned}
\partial_t \epsilon &= -\partial_i J^i_\epsilon + \mathcal{O}(\partial_i)^{2n+3}\,, \\
\partial_t p_i &= -\partial_j T^{ji} + \mathcal{O}(\partial_i)^{2n+2}\,, \\
\partial_t n &= -\partial_i \partial_j J^{ij} + \mathcal{O}(\partial_i)^{2n+3}\,.
\end{aligned}$$

Notice that onshell temporal derivatives will not be independent from spatial gradients, in particular we have the hierarchy $\mathcal{O}(\partial_t) \sim \mathcal{O}(\partial_i)^2$.

Constitutive relations

Entropy current

$$\partial_i S^i - \frac{1}{T}\partial_i \mathcal{E}^i + \frac{\tilde{\mu}}{T}\partial_i\partial_j J^{ij} + \frac{V_i}{T}\partial_j T^{ji} \geq 0\,.$$

Using this we can fix the form of constitutive relations

$$J^i_\epsilon = (\epsilon + P)V_i - F_{ij}\partial_t(\frac{p_j}{n}) + \alpha\partial_i\frac{1}{T}\,,$$

$$J^{ij} = -F_{ij}\,,$$

$$T^{ij} = P\delta_{ij} + V_i p_j + V_j p_i + \partial_k F_{ij}\frac{p_k}{n} + F_{ij}V_{kk}\,,$$

where $J^i_\epsilon = \mathcal{E}^i - F_{ij}\partial_t\left(\frac{p_j}{n}\right)$.

We get one dissipative coefficient at this level of the expansion.

Towards fracton superfluids

The state-of-the-art developments in fracton superfluidity involves a first steps in the systematic study of symmetry breaking patterns and dissipation.

P-Wave: dipole broken, U(1) unbroken
S-wave - Both dipole and U(1) broken

The challenge is to correctly develop a gradient expansion.

Jensen, Raz

Glorioso et al.

Armas, Have

Jain et al.

Głódkowski et al.

Conclusions

- A lot of progress in understand hydrodynamic and quasi-hydrodynamic theories with intrinsic symmetries
- Lessons from elasticity theory
- New examples of spontaneous symmetry breaking
- New insights into defect dynamics

Thank you!